An unsure talk on an un-Schur problem

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Results are joint work with...

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Monochromatic Schur triples and Ramsey properties

Theorem (Schur 1916)

Every c-coloring of $[n]$ contains a monochromatic Schur triple for $n \geq n_0(c)$.

Graham, Rödl, and Ruciński (1996) showed that asymptotically at least an 0.04 proportion needs to be monochromatics for $c = 2$ using a result of Goodman (1959).

Question (Chen and Graham at SOCA 96, \$100).

Is the proportion $2/11 = 0.\overline{18}$ given by a construction of Zeilberger optimal?

Theorem (Datskovsky, 2003; Schoen, 1999; Robertson and Zeilberger, 1998)

Yes, and the construction is unique!

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Rainbow Schur triples and anti-Ramsey properties

Theorem (Alekseev and Savchev, 1987; Schönheim, 1990)

Every 3-coloring of $[n]$ needs to contain an un-Schur triple, i.e., a rainbow Schur triple, for large enough n as long as each color class covers at least a $1/4$ proportion of $[n]$.

Question. What is the maximum proportion of Schur triples that can be rainbow?

This (surprisingly) seems to not have been explicitly asked before, even though the graph theory equivalent due to Erdős and Sós was resolved by Balogh et al. (2017).

Theorem (Parczyk and S., 2024+)

At least 0.4 and at most 0.66656. We conjecture the lower bound to be tight.

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Two obvious candidates

Intervals. $c_{\text{int}} : i \mapsto \begin{cases} \text{blue} & \text{if } 1 \leq i < n/3 \\ \text{red} & \text{if } n/3 \leq i < 2n/3 \\ \text{green} & \text{if } 2n/3 \leq i \leq n \end{cases}$ gives $2/9 = 0.\overline{2}$.

Modulus. $c_{\text{mod}} : i \mapsto \begin{cases} \text{red} & \text{if } i \equiv 1 \mod 4 \\ \text{blue} & \text{if } i \equiv 2 \mod 4 \\ \text{green} & \text{if } i \equiv 3 \mod 4 \\ \text{blue} & \text{if } i \equiv 0 \mod 4 \end{cases}$ gives $3/8 = 0.375$.

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The best construction

However, the best construction gives $2/5 = 0.4$ by combining both ideas

$$c^* : i \mapsto \begin{cases} \text{blue} & \text{if } i \text{ is odd and } i \leq 2n/5 \\ \text{red} & \text{if } i \text{ is odd and } i > 2n/5 \\ \text{green} & \text{if } i \text{ is even} \end{cases}.$$

It was found by looking at the (unique) best solutions for $n = 10, 11, 12, \ldots, 30$ obtained by solving a MILP formulation of the underlying problem.

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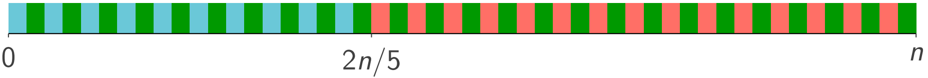

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Fixing some notation

Schur triples. Let $S := \{(x, y, z) \in [n]^3 \mid x + y = z\}$ and $S(z) := \{(\cdot, \cdot, z) \in S\}$, that is for us Schur triples are *ordered* and *not-necessarily distinct*. Clearly

$$|S| = \binom{n}{2} \quad \text{and} \quad |S(z)| = z - 1. \tag{1}$$

Colorings. For any coloring $c : [n] \to \{1, 2, 3\}$, write $\mathrm{r}_c(z)$ for the number of Schur triples from $S(z)$ that are rainbow under c, so we are interested in an upper bound on

$$\sum_{z=1}^{n} \mathrm{r}_c(z).$$

Let's obtain an upper bound through a result in graph theory...

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Relating Schur triples to triangles

Let $T = \{(v_1, v_2, v_3) \mid v_1 < v_2 < v_3 \in [n+1]\}$ denote the set of all triangles in K_{n+1} and consider the map $f : T \to S$, $(v_1, v_2, v_3) \mapsto (v_2 - v_1, v_3 - v_2, v_3 - v_1)$. Let c induce a coloring

$$c' : E(K_{n+1}) \to \{1, 2, 3\}, \quad e \mapsto c(\max e - \min e).$$

Any Schur triple in $S(z)$ exactly corresponds to $n + 1 - z$ triangles and any triangle is rainbow if the underlying Schur triple is rainbow, so by Balogh et al. (2017)

$$\sum_{z=1}^{n} \mathrm{r}_c(z)\,(n+1-z) \leq (1/15 + o(1))\, n^3. \tag{2}$$

Knowing the upper bound on $\sum_{z=1}^{n} \mathrm{r}_c(z)\,(n+1-z)$, we want one for $\sum_{z=1}^{n} \mathrm{r}_c(z)$.

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A reweighting lemma

Lemma

For any finite set S, functions $f, g : S \to \mathbb{R}_{\geq 0}$, and $S_0 \subseteq S$ and $f_0 : S_0 \to \mathbb{R}$ satisfying

(i) $g|_{S \setminus S_0} > 0$

(ii) $\max g|_{S_0} \leq \min g|_{S \setminus S_0}$,

(iii) $f|_{S_0} \leq f_0$, and

(iv) $\sum_{s \in S_0} f_0(s)\, g(s) \geq \sum_{s \in S} f(s)\, g(s)$,

we have $\sum_{s \in S} f(s) \leq \sum_{s \in S_0} f_0(s)$.

An easy bound. Let us first follow the ideas of Graham, Rödl, Ruciński (1996) by choosing $S = [n]$, $f(z) = \mathrm{r}_c(z)$, $g(z) = n + 1 - z$, $f_0(z) = z - 1$, and $S_0 = \{z \mid z \geq z_0\}$, where $z_0 = z_0(n) \in [n]$ is chosen maximal such that

$$\sum_{z=1}^{n} \mathrm{r}_c(z)\,(n + 1 - z) \leq \sum_{z=z_0}^{n} (z - 1)\,(n + 1 - z). \tag{3}$$

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An easy bound

Writing $\alpha = \alpha(n) = z_0(n)/n$, the lemma tells us that

$$\sum_{z=1}^{n} \mathrm{r}_c(z) \leq \sum_{z=z_0}^{n} z - 1 = \left(1/2 - \alpha^2/2 + o(1)\right) n^2. \tag{4}$$

By maximality of z_0 and by (2), we also have

$$\left(1/15 + o(1)\right) n^3 \geq \sum_{z=1}^{n} \mathrm{r}_c(z) \left(n + 1 - z\right) > \left(1/6 - \alpha^2/2 + \alpha^3/3 + o(1)\right) n^3,$$

and therefore $\sum_{z=1}^{n} \mathrm{r}_c(z) \leq \left(0.33922 + o(1)\right) n^2$ by (4).

A slightly less easy bound. We assumed that all triples in $S(z)$ are rainbow if $z \geq z_0$. Taking this into account, one can push the bound slightly to 0.33328.

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Why isn't the bound better?

Two clear avenues for improvement:

1. Choice of z_0 is suboptimal. It neither accounts for the partially modular nature of the extremal construction nor captures the linear dependence of $\mathrm{r}_c(z)$ on z.

2. The Balogh et al. bound is not tight for our setting. The family of induced edge-colorings look very different from the known extremal coloring.

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What about arithmetic progressions?

Related question 1. What's the maximum proportion of k-term arithmetic progressions that can be rainbow? The case $k = 3$ was remarked to be $2/3$ by Jungić et al. (2003). This can easily be extended to show that The maximum fraction is at least

$$\prod_{i=1}^{m}(1 - 1/p_i) \quad \text{if} \quad k = p_1^{a_1} \cdots p_m^{a_m}.$$

by coloring mod $p_1 \cdots p_m$ and at most $(1 - 1/k)$. We again conjecture the lower bound to be tight.

Related question 2. On can also consider similar questions in $\mathbb{Z}_n$ instead of $[n]$, which made the problem easier for monochromatic Schur triples, though here the optimal constructions seems to depend on the precise prime decomposition of n.

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Thank you for your attention!

Preprint is available at `arxiv.org/abs/2410.22024`.