The Rado Multiplicity Problem in $\mathbb{F}_q^n$

Eurocomb 2023 at Charles University

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Given a **coloring** $\gamma : \mathbb{F}_q^n \to [c]$ and linear map L, we are interested in

$$\mathcal{S}_L(\gamma) \overset{\text{def}}{=} \{\mathbf{s} \in (\mathbb{F}_q^n)^m : L(\mathbf{s}) = \mathbf{0}, s_i \neq s_j \text{ for } i \neq j, \mathbf{s} \in \gamma^{-1}(\{i\})^m \text{ for some } i\}. \tag{1}$$

Rado (1933) tells us that $\mathcal{S}_L(\gamma) \neq \emptyset$ for large enough n if L satisfies *column condition*.

The **Rado Multiplicity Problem** is concerned with determining

$$m_{q,c}(L) \overset{\text{def}}{=} \lim_{n\to\infty} \min_{\gamma \in \Gamma(n)} |\mathcal{S}_L(\gamma)| \,/\, |\mathcal{S}_L(\mathbb{F}_q^n)|.$$

Limit exists by monotonicity and $0 < m_{q,c}(L) \leq 1$ if L is partition regular. L is c-**common** if $m_{q,c}(L) = c^{1-m}$ (the value attained in a uniform random coloring).

This reminds us of an old question of Erdős in graph theory ...

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History of the problem

- Graham et al. (1996) gave lower bound for **Schur triples** in 2-colorings of $[n]$, later independently resolved by Robertson and Zeilberger / Schoen / Datskovsky.
- Cameron et al. (2007) showed that the nr. of solutions for linear equations with **an odd nr. of variables** only depends on cardinalities of the two color classes.
- Parrilo, Robertson and Saracino (2008) established bounds for the minimum number of **monochromatic** 3-**APs** in 2-colorings of $[n]$ (not 2-common in $\mathbb{N}$).
- For $r = 1$ and m even, Saad and Wolf (2017) showed that any 'pair-partitionable' L is 2-common in $\mathbb{F}_q^n$. Fox, Pham, and Zhao (2021) showed that this is necessary and Versteegen further generalized their result.
- Kamčev et al. (2021) characterized some non-common L in $\mathbb{F}_q^n$ with $r > 1$.
- Král et al. (2022) characterized 2-common L for $q = 2$, $r = 2$, m odd.

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Our results

We are interested in particular L and $\mathbb{F}_q^n$.

Theorem (Rué and S., 2023)

We have $1/10 < m_{q=5,c=2}(L_{4\text{-}AP}) \leq 0.1\overline{031746}$.

Saad and Wolf (2017) previously established an u.b. of 0.1247 with no no-trivial l.b. known.

Proposition (Rué and S., 2023)

We have $m_{q=3,c=3}(L_{3\text{-}AP}) = 1/27$.

Similar to Cummings et al. (2013) extending a result of Goodman (1959) about triangles.

Upper bounds
obtained through blow-up constructions of particular finite colorings.

Lower bounds
obtained by extending Razborov's Flag Algebra framework.

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How to blow up colorings

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The 'quantitatively superior' approach however is to consider *blowups*:

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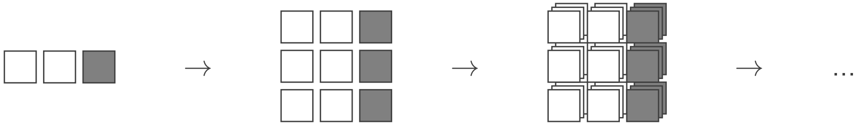

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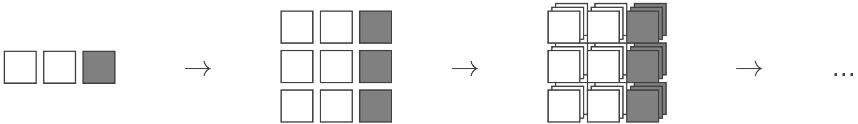

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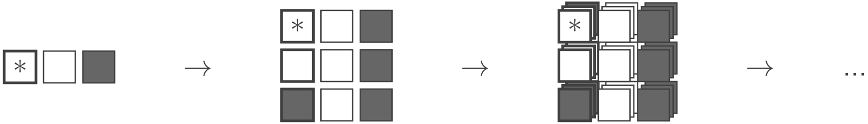
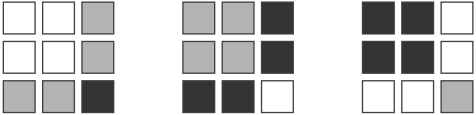


Proofs of the upper bounds

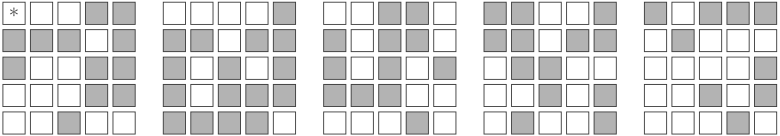
Upper bound of the Theorem

$m_{3,3}(L_{4\text{-AP}}) \leq 1/27$ follows from the blow-up of this 3-coloring of $\mathbb{F}_3^3$:



Upper bound of the Proposition

$m_{5,2}(L_{4\text{-AP}}) \leq 13/126$ follows from the iterated blow-up of this 2-coloring of $\mathbb{F}_5^3$:



Where are Flag Algebras in Additive Combinatorics?

In the table below we have marked in bold a monochromatic or rainbow arithmetic progression in each 3-coloring of the 9-tuples. This proves that any 3-coloring of any 9-tuple contains a non-degenerate arithmetic progression of length 3 belonging to M or R.

$\mathbf{111}\ast\ast\ast\ast\ast\ast$	$1\mathbf{1}22\mathbf{1}22\mathbf{1}\ast$	$12122\mathbf{111}\ast$	$1221\mathbf{213}\ast\ast$
$112\mathbf{111}\ast\ast\ast$	$11221\mathbf{222}\ast$	$1\mathbf{2}12\mathbf{2}11\mathbf{2}\ast$	$\mathbf{1}22\mathbf{1}22\mathbf{1}\ast\ast$
$\mathbf{1}12\mathbf{1}12\mathbf{1}\ast\ast$	$112212\mathbf{231}$	$121\mathbf{2}2\mathbf{1}1\mathbf{3}\ast$	$1221\mathbf{222}\ast\ast$
$1\mathbf{1}21\mathbf{1}22\mathbf{1}\ast$	$11\mathbf{2}21\mathbf{2}23\mathbf{2}$	$12\mathbf{1}22\mathbf{1}21\mathbf{1}$	$12212\mathbf{231}\ast$
$11211\mathbf{222}\ast$	$1122\mathbf{1}2\mathbf{2}3\mathbf{3}$	$1212\mathbf{2}1\mathbf{2}1\mathbf{2}$	$1\mathbf{2}21\mathbf{2}23\mathbf{2}\ast$
$112\mathbf{1}1\mathbf{2}2\mathbf{3}\ast$	$1122\mathbf{123}\ast\ast$	$121221\mathbf{213}$	$122\mathbf{1}2\mathbf{2}3\mathbf{3}\ast$
$1121\mathbf{123}\ast\ast$	$112\mathbf{213}\ast\ast\ast$	$1\mathbf{2}12\mathbf{2}12\mathbf{2}\ast$	$122\mathbf{123}\ast\ast\ast$
$\mathbf{1}12\mathbf{1}13\mathbf{1}\ast\ast$	$11\mathbf{222}\ast\ast\ast\ast$	$12122\mathbf{123}\ast$	$12\mathbf{213}\ast\ast\ast\ast$
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$1\mathbf{1}2\mathbf{1}2\mathbf{1}\ast\ast\ast$	$12\mathbf{111}\ast\ast\ast\ast$	$1212\mathbf{231}\ast\ast$	$122\mathbf{321}\ast\ast\ast$
$\mathbf{1}12\mathbf{1}22\mathbf{1}\ast\ast$	$\mathbf{1}21\mathbf{1}21\mathbf{1}\ast\ast$	$12122\mathbf{321}\ast$	$122\mathbf{3}2\mathbf{2}1\mathbf{1}\ast$
[illegible]	[illegible]	[illegible]	[illegible]

Figure: Cameron, Peter J., Javier Cilleruelo, and Oriol Serra. "On monochromatic solutions of equations in groups." Revista Matemática Iberoamericana 23.1 (2007): 385-395.

Correctly defining our combinatorial structures

Definition (Partially fixed Morphisms, Monomorphisms, and Isomorphisms)

An affine linear map $\varphi : \mathbb{F}_q^k \to \mathbb{F}_q^n$ as a *t-fixed morphism* iff $\varphi(e_j) = e_j$ for all $0 \leq j \leq t$ (where $t \geq -1$ and $e_0 = 0$). It is a *mono/isomorphism* iff it is in/bijective.

This gives us ...

- ... a notion of isomorphic colorings through isomorphisms,
- ... a notion of substructure or sub-coloring through monomorphisms,
- ... a notion of density,
- ... a notion of a 'type' through t.

The resulting notion of density crucially satisfies the *averaging equality*

$$p(\texttt{small}, \texttt{large}) = \sum_{\texttt{medium}} p(\texttt{small}, \texttt{medium}) \cdot p(\texttt{medium}, \texttt{large}). \tag{2}$$

Correctly defining solutions

Problem. How to count solutions through colorings? In $\mathbb{F}_3^n$ for example, the Schur triple $(0,0,\overline{0}),(1,2,\overline{0}),(2,1,\overline{0})$ defines a unique 2-dimensional linear subspace, but the Schur triple $(0,0,\overline{0}),(1,1,\overline{0}),(2,2,\overline{0})$ does not ...

Definition

The *dimension* $\dim_t(\mathbf{s})$ of $\mathbf{s} \in \mathcal{S}_L$ is the smallest dimension of a t-fixed subspace containing it and $\dim_t(L)$ is the largest dimension of any solution.

Each fully dimensional solution determines a unique $\dim_t(L)$-dimensional substructure in which it lies. Writing $\mathcal{S}_L^t(T) = \{\mathbf{s} \in \mathcal{S}_L(T) : \dim_t(\mathbf{s}) = \dim_t)(L)\}$, we have

$$|\mathcal{S}_L^t(\mathbb{F}_q^n)| = |\mathcal{S}(\mathbb{F}_q^n)|\,(1 + o(1)).$$

Fully-dimensional solution satisfy an averaging equality like (2)**.**

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SOS please someone help me

Definition

The *flag algebra* $\mathcal{A}$ is given by considering linear combinations of colorings, factoring out relations given by the averaging equality (2) and defining an appropriate product.

Razborov established a bijection between sequences (G_n) where all $p(H; G_n)$ converge and $\varphi \in \mathrm{Hom}(\mathcal{A}, \mathbb{R})$ satisfying $\varphi(H) \geq 0$ for all $H \in \mathcal{G}$ through $p(H; G_n) = \varphi(H)$.

The *semantic cone* $\mathcal{S} = \{f \in \mathcal{A} : \phi(f) \geq 0$ for all $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})\}$ captures those algebraic expressions that correspond to density expressions that are 'true'.

Letting $C_L \in \mathcal{A}$ capturing the behavior of $|\mathcal{S}_L^t(\mathbb{F}_q^n)|$, we can establish a lower bound by finding and verifying a sum-of-squares (SOS) expression

$$C_L - \lambda - \sum_{i=1}^{k} f_i^2 \in \mathcal{S}. \tag{3}$$

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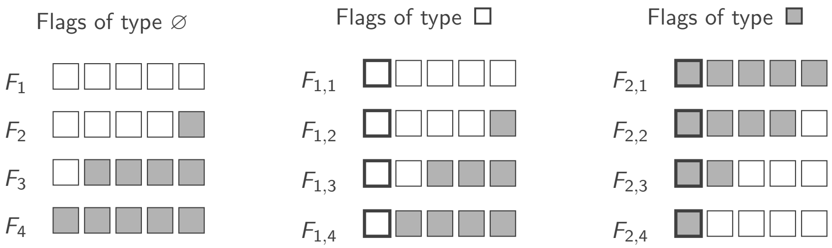
$$C_L - \lambda - \sum_{i=1}^{k} f_i^2 \in \mathcal{S}. \tag{3}$$

Lower bound of the Proposition

$m_{5,2}(L_{\text{4-AP}}) > 1/10$ follows by verifying that over all 3324 2-colorings of $\mathbb{F}_5^2$ we have

$$F_1 + F_4 + (F_2 + F_3)/5 - 1/10 \geq \sum_{i=1}^{2} \Big(9/10 \cdot \big[\!\big[(F_{i,1} + (5\,F_{i,2} - 5\,F_{i,3} - 10\,F_{i,4})/27)^2 \big]\!\big]_{-1}$$
$$\ldots + 61/162 \cdot \big[\!\big[((F_{i,3} - F_{i,2})/2 + F_{i,4})^2 \big]\!\big]_{-1} \Big),$$

and by noting that $F_{1,1} + F_{2,1} > 0$. Here the relevant flags F_i and $F_{i,j}$ are

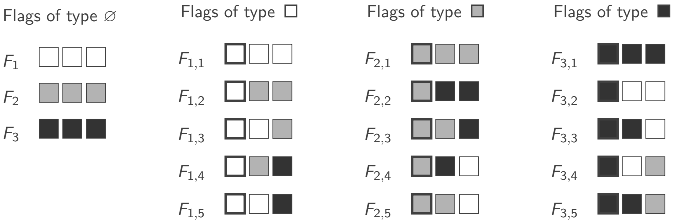


Lower bound of the Theorem

$m_{3,3}(L_{\text{3-AP}}) \geq 1/27$ follows by verifying that over all all 140 3-colorings of $\mathbb{F}_3^2$ we have

$$
\begin{aligned}
F_i - 1/27 \geq 26/27 \cdot & \left[\!\left[\left(F_{i,1} - 99/182\, F_{i,2} + 75/208\, F_{i,3} - 11/28\, F_{i,4} - 3/26\, F_{i,5}\right)^2 \right]\!\right]_{-1} \\
& \ldots + 1685/1911 \cdot \left[\!\left[\left(F_{i,2} - 231/26960\, F_{i,3} + 1703/6740\, F_{i,4} - 1869/3370\, F_{i,5}\right)^2 \right]\!\right]_{-1} \\
& \ldots + 71779/431360 \cdot \left[\!\left[\left(F_{i,3} - 358196/502453\, F_{i,4} - 412904/502453\, F_{i,5}\right)^2 \right]\!\right]_{-1} \\
& \ldots + 5431408/10551513 \cdot \left[\!\left[\left(F_{i,4} - 1/4\, F_{i,5}\right)^2 \right]\!\right]_{-1}
\end{aligned}
$$

for any $i \in \{1, 2, 3\}$. Here the relevant flags F_i and $F_{i,j}$ are



Open problems and final remarks

- Often one can extract stability results from Flag Algebra certificates.
- Steep computational hurdle: underlying structures grow exponentially (instead of quadratically for graphs or cubic for 3-uniform hypergraphs)
- No neat notion of subspaces makes generalizing to other groups difficult.

Code is available at `github.com/FordUniver/rs_radomult_23`

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Thank you for your attention!