Towards Flag Algebras in Additive Combinatorics

FoCM 2023 – Workshop I.3:
Graph Theory and Combinatorics

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The Motivation

Why are there no Flag Algebras in Additive Combinatorics?

Given $T \subset G$ and linear map L, we care about

$$\mathcal{S}_L(T) \stackrel{\text{def}}{=} \{\mathbf{s} \in T^m : L(\mathbf{s}) = \bar{0}, s_i \neq s_j \text{ for } i \neq j\}, \quad (1)$$

where $G = [n]$, $\mathbb{Z}_n$, $\mathbb{Z}_p$, $\mathbb{F}_q^n$, $\ldots$ and L represents AP, Schur triples, ...

- Ramsey (1930) $\leftrightsquigarrow$ Schur (1917), van der Waerden (1927), and Rado (1933)
- Mantel (1907) and Turán (1941) $\leftrightsquigarrow$ Roth (1953) and Szémeredi (1975)
- regularity lemma (Szémeredi, 1978) $\leftrightsquigarrow$ arithmetic regularity (Green, 2005)
- random graph $G(n, p)$ $\leftrightsquigarrow$ random sets $[n]_p$, $(\mathbb{Z}_n)_p$, $\ldots$
- blowup-type constructions are relevant in both

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In the table below we have marked in bold a monochromatic or rainbow arithmetic progression in each 3-coloring of the 9-tuples. This proves that any 3-coloring of any 9-tuple contains a non-degenerate arithmetic progression of length 3 belonging to M or R.

$\mathbf{111}******$	$1\mathbf{1}22\mathbf{1}22\mathbf{1}*$	$12122\mathbf{111}*$	$1221\mathbf{213}**$
$112\mathbf{111}***$	$11221\mathbf{222}*$	$1\mathbf{2}12\mathbf{2}11\mathbf{2}*$	$\mathbf{1}22\mathbf{1}22\mathbf{1}**$
$\mathbf{1}12\mathbf{1}12\mathbf{1}**$	$112212\mathbf{231}$	$121\mathbf{2}2\mathbf{1}1\mathbf{3}*$	$1221\mathbf{222}**$
$1\mathbf{1}21\mathbf{1}22\mathbf{1}*$	$11\mathbf{2}21\mathbf{2}23\mathbf{2}$	$12\mathbf{1}22\mathbf{1}21\mathbf{1}$	$12212\mathbf{231}*$
$11211\mathbf{222}*$	$1122\mathbf{1}2\mathbf{2}3\mathbf{3}$	$1212\mathbf{2}1\mathbf{2}1\mathbf{2}$	$1\mathbf{2}21\mathbf{2}23\mathbf{2}*$
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[illegible]	[illegible]	[illegible]	[illegible]

Figure: Cameron, Peter J., Javier Cilleruelo, and Oriol Serra. "On monochromatic solutions of equations in groups." Revista Matemática Iberoamericana 23.1 (2007): 385-395.

A Problem

What we need is a rule like

$$p(\texttt{small}, \texttt{large}) = \sum_{\texttt{medium}} p(\texttt{small}, \texttt{medium}) \cdot p(\texttt{medium}, \texttt{large}) \qquad (2)$$

for some notion of density

$$p(\texttt{struct}, \bullet) = \frac{\#\text{ substructures isomorphic to } \texttt{struct} \text{ in } \bullet}{\#\text{ substructures of same size as } \texttt{struct} \text{ in } \bullet}. \qquad (3)$$

Finding a working notion of substructure seems difficult in $[n]$, $\mathbb{Z}_n$, $\mathbb{Z}_p$ …

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Counting monochromatic solutions

Given a **coloring** $\gamma : \mathbb{F}_q^n \to [c]$ and linear map, we are interested in

$$\mathcal{S}_L(\gamma) \stackrel{\text{def}}{=} \bigcup_{i=1}^{c} \mathcal{S}_L(\gamma^{-1}(\{i\})). \tag{4}$$

Rado (1933) tells us that $\mathcal{S}_L(\gamma) \neq \emptyset$ for large enough n if L satisfies *column condition*.

The **Rado Multiplicity Problem** is concerned with determining

$$m_{q,c}(L) \stackrel{\text{def}}{=} \lim_{n\to\infty} \min_{\gamma\in\Gamma(n)} |\mathcal{S}_L(\gamma)| \,/\, |\mathcal{S}_L(\mathbb{F}_q^n)|.$$

Limit exists by monotonicity and $0 < m_{q,c}(L) \leq 1$ if L is partition regular. L is c-**common** if $m_{q,c}(L) = c^{1-m}$ (the value attained in a uniform random coloring).

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Previous results

- Graham et al. (1996) gave lower bound for **Schur triples** in 2-colorings of $[n]$, later independently resolved by Robertson and Zeilberger / Schoen / Datskovsky.
- Cameron et al. (2007) showed that the nr. of solutions for linear equations with **an odd nr. of variables** only depends on cardinalities of the two color classes.
- Parrilo, Robertson and Saracino (2008) established bounds for the minimum number of **monochromatic** 3-**APs** in 2-colorings of $[n]$ (not 2-common in $\mathbb{N}$).
- For $r = 1$ and m even, Saad and Wolf (2017) showed that any 'pair-partitionable' L is 2-common in $\mathbb{F}_q^n$. Fox, Pham, and Zhao (2021) showed that this is necessary.
- Kamčev et al. (2021) characterized some non-common L in $\mathbb{F}_q^n$ with $r > 1$.
- Král et al. (2022) characterized 2-common L for $q = 2$, $r = 2$, m odd.

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Our results

Theorem (Rué and S., 2023)

We have $1/10 < m_{q=5,c=2}(L_{4\text{-}AP}) \leq 0.1\overline{031746}$.

Saad and Wolf (2017) previously established an u.b. of 0.1247 with no no-trivial l.b. known.

Proposition (Rué and S., 2023)

We have $m_{q=3,c=3}(L_{3\text{-}AP}) = 1/27$.

Similar to Cummings et al. (2013) extending a result of Goodman (1959) about triangles.

Proofs are computational:

- **Upper bounds** obtained through (iterated) blow-up constructions found through exhaustive and heuristic searches.
- **Lower bounds** obtained through SOS expressions in Flag Algebras found through an SDP solver.

The Right Notion of Substructure

Definition (Partially fixed Morphisms, Monomorphisms, and Isomorphisms)

An affine linear map $\varphi : \mathbb{F}_q^k \to \mathbb{F}_q^n$ as a *t-fixed morphism* iff $\varphi(e_j) = e_j$ for all $0 \leq j \leq t$ (where $t \geq -1$ and $e_0 = 0$). It is a *mono/isomorphism* iff it is in/bijective.

This gives us ...

- ... a notion of isomorphic colorings through isomorphisms,
- ... a notion of substructure or sub-coloring through monomorphisms,
- ... a notion of density through (3) that satisfies (2),
- ... blow-up bounds through not-necessarily-injective morphisms,
- ... a notion of a 'type' through t,

Remark

The 'base' case is $t = -1$ for invariant structures and $t = 0$ otherwise.

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Counting solutions through colorings

Problem. How to count solutions through colorings? In $\mathbb{F}_3^n$ for example, the Schur triple $(0,0,\overline{0}),(1,2,\overline{0}),(2,1,\overline{0})$ defines a unique 2-dimensional linear subspace, but the Schur triple $(0,0,\overline{0}),(1,1,\overline{0}),(2,2,\overline{0})$ does not ...

Definition

The *dimension* $\dim_t(\mathbf{s})$ of $\mathbf{s} \in \mathcal{S}_L$ is the smallest dimension of a t-fixed subspace containing it and $\dim_t(L)$ is the largest dimension of any solution.

Each fully dimensional solution determines a unique $\dim_t(L)$-dimensional substructure in which it lies. Writing $\mathcal{S}_L^t(T) = \{\mathbf{s} \in \mathcal{S}_L(T) : \dim_t(\mathbf{s}) = \dim_t)(L)\}$, we have

$$|\mathcal{S}_L^t(\mathbb{F}_q^n)| = |\mathcal{S}(\mathbb{F}_q^n)|\,(1 + o(1)).$$

So fully-dimensional solutions is what we are *actually* counting!

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SOS please someone help me

Definition

The (unfixed or 0-fixed) *flag algebra* $\mathcal{A}$ is given by considering linear combinations of (unfixed or 0-fixed) colorings, factoring out (2) and defining an appropriate product.

The *semantic cone*

$$\mathcal{S} = \{f \in \mathcal{A} : \phi(f) \geq 0 \text{ for all } \phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})\} \tag{5}$$

captures those algebraic expressions corresponding to density expressions that are 'true'. We can establish a lower bound through

$$C_L - \lambda - \sum_{i=1}^{k} (f_i)^2 \in \mathcal{S}, \tag{6}$$

where $C_L \in \mathcal{A}$ counts fully-dimensional solutions. Such sum-of-squares (SOS) expressions are solvable through Semidefinite Programming (SDP).

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Lower bound of the Proposition

$m_{5,2}(L_{4\text{-AP}}) > 1/10$ follows by verifying that

$$F_1 + F_4 + (F_2 + F_3)/5 - 1/10 \geq \sum_{i=1}^{2} \Big(9/10 \cdot \big[\!\big[(F_{i,1} + (5\,F_{i,2} - 5\,F_{i,3} - 10\,F_{i,4})/27)^2\big]\!\big]_{-1}$$
$$\ldots + 61/162 \cdot \big[\!\big[((F_{i,3} - F_{i,2})/2 + F_{i,4})^2\big]\!\big]_{-1}\Big),$$

over all 3324 2-colorings of $\mathbb{F}_5^2$ (and by noting that $F_{1,1} + F_{2,1} > 0$), where

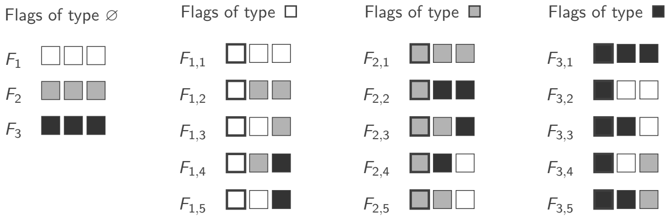
Flags of type ∅		Flags of type □		Flags of type ■	
F_1	□□□□□	$F_{1,1}$	□□□□□	$F_{2,1}$	■■■■■
F_2	□□□□■	$F_{1,2}$	□□□□■	$F_{2,2}$	■■■■□
F_3	□■■■■	$F_{1,3}$	□□■■■	$F_{2,3}$	■■□□□
F_4	■■■■■	$F_{1,4}$	□■■■■	$F_{2,4}$	■□□□□

Lower bound of the Theorem

$m_{3,3}(L_{3\text{-AP}}) \geq 1/27$ follows by verifying that

$$F_i - 1/27 \geq 26/27 \cdot \left[\!\left[\left(F_{i,1} - 99/182\, F_{i,2} + 75/208\, F_{i,3} - 11/28\, F_{i,4} - 3/26\, F_{i,5}\right)^2 \right]\!\right]_{-1}$$
$$\ldots + 1685/1911 \cdot \left[\!\left[\left(F_{i,2} - 231/26960\, F_{i,3} + 1703/6740\, F_{i,4} - 1869/3370\, F_{i,5}\right)^2 \right]\!\right]_{-1}$$
$$\ldots + 71779/431360 \cdot \left[\!\left[\left(F_{i,3} - 358196/502453\, F_{i,4} - 412904/502453\, F_{i,5}\right)^2 \right]\!\right]_{-1}$$
$$\ldots + 5431408/10551513 \cdot \left[\!\left[\left(F_{i,4} - 1/4\, F_{i,5}\right)^2 \right]\!\right]_{-1}$$

for any $i \in \{1, 2, 3\}$ over all 140 3-colorings of $\mathbb{F}_3^2$, where



Final Remarks

- Often one can extract stability results from Flag Algebra certificates.
- Steep computational hurdle: underlying structures grow exponentially
- No neat notion of subspaces makes generalizing to $[n] \,/\, \mathbb{Z}_n \,/\, \mathbb{Z}_p$ difficult.

Code is available at `github.com/FordUniver/rs_radomult_23`

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Thank you for your attention!

How many colorings are there?

q/n	1	2	3	4	5
2	3	5	10	32	382
3	4	14	1028		
4	8	1648			
5	6	3324			

q/n	1	2	3	4	5
2	4	8	20	92	2744
3	6	36	15 636		
4	14	7724			
5	12	72 192			

Table: Number of 2-colorings of $\mathbb{F}_q^n$ up to unfixed (left) and 0-fixed (right) isomorphism.

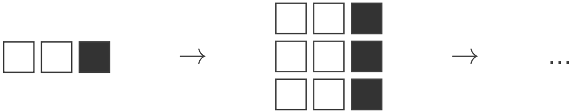
q/n	1	2	3	4
2	6	15	60	996
3	10	140	25 665 178	
4	30	1 630 868		
5	24	70 793 574		

q/n	1	2	3	4
2	9	30	180	6546
3	18	648		
4	69	8 451 708		
5	72			

Table: Number of 3-colorings of $\mathbb{F}_q^n$ up to unfixed (left) and 0-fixed (right) isomorphism.

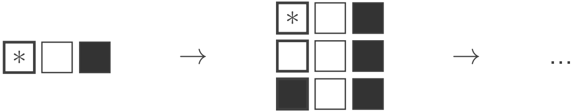
How to blow up colorings

We can *blow up* an colorings into a sequence of colorings with n tending to infinity.



Computing the density of solutions in the limit of this sequence is easy: simply check *not-necessarily-injective* subcolorings in the base construction. **This gives us an immediate upper bound from *any* coloring we can come up with ...**

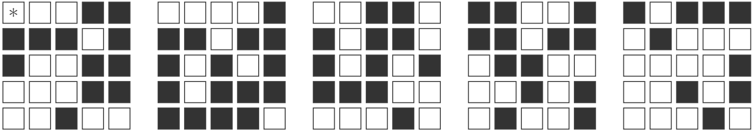
In some cases we have a *free element* in which we can iterate the blowup-construction.



Proofs of the upper bounds

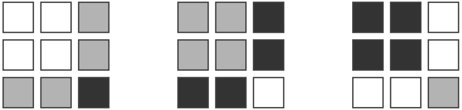
Upper bound of the Proposition

$m_{5,2}(L_{4\text{-AP}}) \leq 13/126$ follows from the iterated blow-up of this 2-coloring of $\mathbb{F}_5^3$:



Upper bound of the Theorem

$m_{3,3}(L_{4\text{-AP}}) \leq 1/27$ follows from the blow-up of this 3-coloring of $\mathbb{F}_3^3$:



Counting Monomorphisms

We write $[n]_q = \sum_{i=0}^{n-1} q^i$ for the *q-number* of n, $[n]_q! = [n]_q \cdots [2]_q\,[1]_q$ for the *q-factorial* of n, and let the *Gaussian multinomial coefficient* be

$$\binom{n}{k_1, \ldots, k_m}_q = \frac{[n]_q!}{[k_1]_q! \,\cdots\, [k_m]_q!\,[n-k']_q!}.$$

Lemma (Double Counting)

We have

$$|\operatorname{Mon}_t(k_1, \ldots, k_m; n')|\,|\operatorname{Mon}_t(n'; n)| = |\operatorname{Mon}_t(k_1, \ldots, k_m; n)| \binom{n-k'}{n'-k'}_q$$

for any $t \geq -1$, $k_1, \ldots, k_m \geq t^+$, *and* $n \geq n' \geq k' = k_1 + \ldots + k_m - (m-1)\,t^+$.

Counting Monomorphisms

Lemma (Unfixed Monomorphisms)

For any integers $0 \leq k_1, \ldots, k_m$ *and* $n \geq k' = k_1 + \ldots + k_m$, *we have*

$$|\operatorname{Mon}_{-1}(k_1, \ldots, k_m; n)| = q^{n-k'} \binom{n}{k_1, \ldots, k_m}_q .$$

Lemma (Fixed Monomorphisms)

For integers $0 \leq t \leq k_1, \ldots, k_m$ *and* $n \geq k' = k_1 + \ldots + k_m - (m-1)\,t$, *we have*

$$|\operatorname{Mon}_t(k_1, \ldots, k_m; n)| = \binom{n-t}{k_1 - t, \ldots, k_m - t}_q .$$