Forcing Graphs to be Forcing

HJSDM 2025 in Tokyo

Aldo Kiem, Olaf Parczyk, **Christoph Spiegel**

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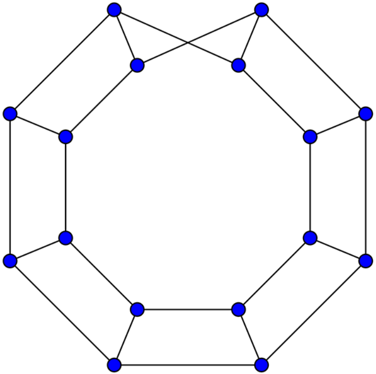

Results are joint work with...

Aldo Kiem
Zuse Institute Berlin
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Zuse Institute Berlin
Freie Universität Berlin

Forcing Graphs to be Forcing

Sidorenko's conjecture

Definition. Let $t(G,H)$ denote the probability that a random map $\phi : V_G \to V_E$ is a graph homomorphism from G to H, i.e., $\{\phi(v), \phi(w)\} \in E_H$ for any $\{v,w\} \in E_G$.

Conjecture (Erdős and Simonovits 1984; Sidorenko 1984 / 1991 / 1993)

Every bipartite graph G satisfies $t(G,H) \geq t(K_2,H)^{e_G}$ for any graph H.

Without guarantee of completeness, graphs known to be Sidorenko include ...

- ... trees, even cycles, complete bipartite graphs (Sidorenko '93), and cubes (Hatami '10)
- ... bipartite graphs with one vertex complete to the other side (Conlon, Fox, Sudakov '10)
- ... cartesian product of a tree with a Sidorenko graph (Kim, Lee, Lee '16)
- ... cartesian product of an even cycle with a Sidorenko graph (Conlon, Kim, Lee, Lee '18)
- ... certain tree like graphs (Szegedy '11 and Kim, Lee, Lee '16)
- ... strongly tree-decomposable graphs (Conlon, Kim, Lee, Lee '18)

A recursive procedure (Lee and Szegedy '11 and Szegedy '14) covers the above. Additionally, there is a degree condition (Conlon and Lee '21) and certain subdivisions (Im, Li, Liu '24).

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The forcing conjecture

Definition. A sequence $(H_n)_{n\in\mathbb{N}}$ is *p-quasi-random* if $t(G, H_n) = (1 + o(1))\, p^{e_G}$ for every graph G. Equivalently (Chung, Graham, Wilson '89), we can also only require

$$t(K_2, H_n) = (1 + o(1))\, p \quad textand \quad t(C_4, H_n) = (1 + o(1))\, p^4. \tag{1}$$

Question. Which graphs are *forcing*, i.e., can replace C_4 in (1)?

Conjecture (Skokan and Thoma 2004; Conlon, Fox, and Sudakov 2010)

Every bipartite graph with at least one cycle is forcing.

The forcing conjecture implies Sidorenko's conjecture and has been shown for ...

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Forcing Graphs to be Forcing

Definition of the graph algebra

Definition. Writing $\mathcal{G}_\simeq$ for the family of equivalence classes of graphs and $\mathbb{R}[\mathcal{G}_\simeq]$ for all formal linear combinations of them, we can define a product with unit $\varnothing$ through

$$G \cdot F = \sum_{\substack{H \in \mathcal{G}_{V_G \sqcup V_F} \\ H[V_G]=G \wedge H[V_F]=F}} H.$$

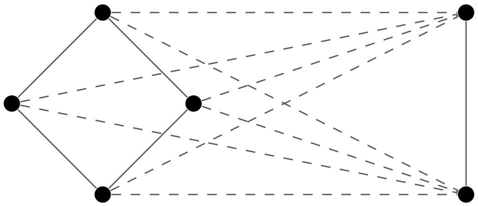
The algebra is obtained through $\mathcal{A} = \mathbb{R}[\mathcal{G}_\simeq]/\langle \varnothing - \bullet \rangle$ with $\bullet$ being a single vertex.

Definition

An element $f \in \mathcal{A}$ is *positive* if it is in the closure of the cone $\mathbb{R}^+[G_\simeq]$, i.e., if $f - \varepsilon$ has a representative with positive coefficients for any ε.

Observation. $f \geq 0$ if and only if $t(f, H) \geq 0$ for all graphs H, so G is Sidorenko iff

$$\mathrm{ni}(G) := \sum_{G \subseteq H} H \geq K_2^{e_G}. \tag{2}$$



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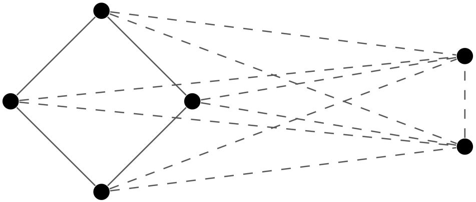

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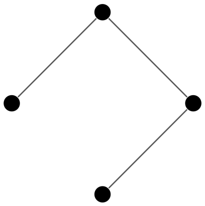
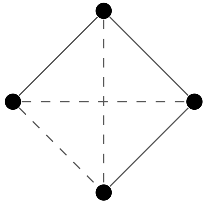


P_3



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Comparison to other graph algebras

	Graph Limits (Lovász, Szegedy 2006)	**Flag Algebras** (Razborov 2007)	**this**
injective?	(mostly) yes	yes	yes
map / subset?	map	subset	map
induced?	(mostly) no	yes	yes
types?	yes	yes	vertex labels
product	non-induced	induced	non-induced
positivity	graphons	positive hom.	cone closure

All of these are effectively equivalent and the following are equal for a given $f \in \mathcal{A}$:

- f is in the closure of $\mathbb{R}^+[G_{\simeq}]$.
- $t(f, H) \geq 0$ for any graph H.
- $\lim_n \mathrm{inj}(f, H_n) \geq 0$ for any convergent sequence $(H_n)_{n\in\mathbb{N}}$.
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Order-preserving operators

Definition (Downward functors and upward transformations)

A function η is a *downward functor* if it maps finite sets to finite sets and finite injections $\alpha : M \hookrightarrow N$ to finite injections $\eta(\alpha) : \eta(M) \hookrightarrow \eta(N)$ s.t.

$$\eta(\alpha) \circ \eta(\beta) = \eta(\alpha \circ \beta) \quad \forall \alpha : N \hookrightarrow T,\ \beta : M \hookrightarrow N. \tag{3}$$

A function $\tau : \mathcal{G}_{\eta(\cdot)} \to \mathcal{G}_{\cdot}$ is an *η-upward transformation* if the following commutes:

$$\tau(\mathrm{ind}_{\eta(\alpha)}\, G) = \mathrm{ind}_{\alpha}\, \tau(G) \quad \forall \alpha : M \hookrightarrow N,\ G \in \mathcal{G}_{\eta(N)}. \tag{4}$$

Remark. Any τ is determined by its restriction to $\mathcal{G}_{\eta(\{0,1\})}$.

Proposition (Kiem, Parczyk, S. 2024+)

The operator given by $[\![G]\!]_{(\eta,\tau)} = \sum_{H \in \mathcal{G}_{\eta(V_G)},\, \tau(H)=G} H$ defines an order-preserving map.

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$$\begin{tikzcd}
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Order-preserving operators (contd.)

Definition (Relevant downward functors)

Let $\eta^{(k)}$ map M to $\binom{M}{k}$ and $\alpha : M \hookrightarrow N$ to the injection mapping $\{m_1, \ldots, m_k\}$ to $\{\alpha(m_1), \ldots, \alpha(m_k)\}$. Given two downward functors η and η', we also let

- ... $\eta \sqcup \eta'$ map M to $\eta(M) \sqcup \eta'(M)$ and α to $\eta(\alpha) \sqcup \eta'(\alpha)$ and
- ... $\eta \times \eta'$ map M to $\eta(M) \times \eta'(M)$ and α to $(m, m') \mapsto (\eta(\alpha)(m), \eta'(\alpha)(m'))$.

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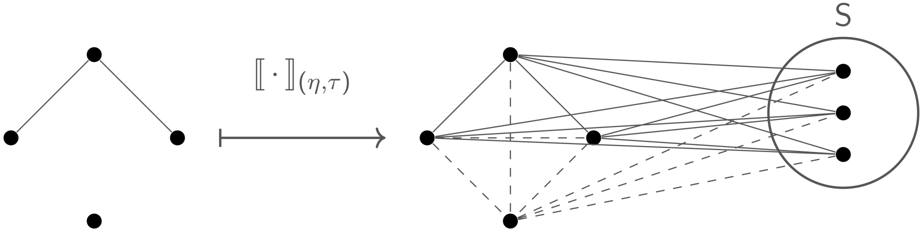
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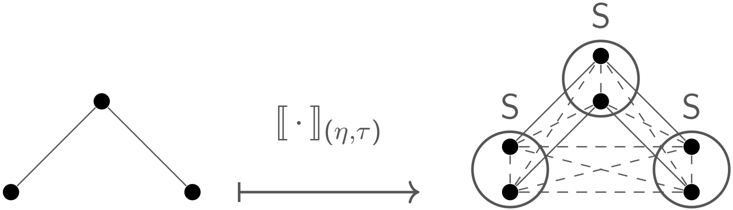
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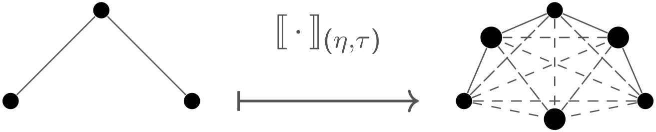
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Forcing Graphs to be Forcing

An example: the tensor power trick

Lemma (Sidorenko 1993)

If $\mathrm{ni}(G) \geq c\, K_2^{e_G}$ *for some* $c > 0$, *then* G *is Sidorenko.*

Proof. Let the downward functor be given by $\eta = \mathrm{id} \times \mathrm{const}_S$ and let τ map supersets of perfect matchings to an edge. We have

$$[\![\mathrm{ni}(G)]\!]_{(\eta,\tau)} = \mathrm{ni}(G)^{|S|} \tag{5}$$

and applying this operator to the assumption $\mathrm{ni}(G) \geq c\, K_2^{e_G}$ therefore gives

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for any S. Letting $|S| \to \infty$, it follows that G is Sidorenko by (2). $\square$

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The m-fold blowup $\mathcal{B}_m(G)$ of a Sidorenko graph G is forcing for $m \geq 2$.

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The subdivision $\mathrm{sub}_{F,s,t}(G)$ of a a Sidorenko graph G by another Sidorenko graph F symmetric w.r.t s and t is Sidorenko. If F is forcing, then so is the subdivision.

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Box product, hypergraphs, forcing pairs, and M_5

Through some additional tools, we can also show that ...

... **loose and even hypergraphs** obtained from a Sidorenko graph are also Sidorenko (though Sidorenko's conjecture does *not* generalize to hypergraphs).

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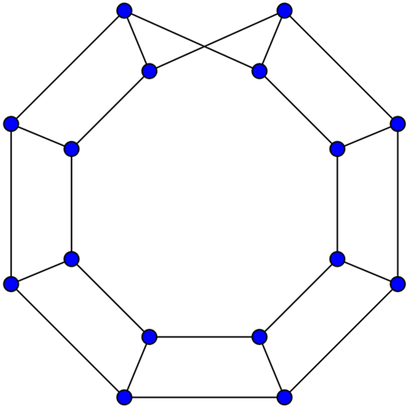

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ご清聴ありがとうございました！

Köszönöm szépen a figyelmet!

Preprint available at `arxiv.org/abs/2412.12904`