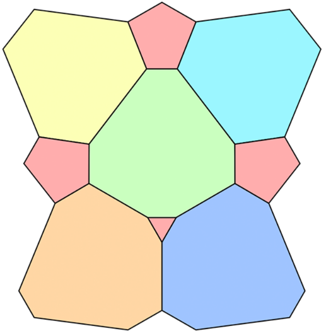
Extending the Continuum of Six-Colorings

KOLKOM 2024 at Heidelberg University

K. Mundinger, S. Pokutta, **C. Spiegel** and M. Zimmer

11th of October 2024



Results are joint work with...

Konrad Mundinger
Zuse Institut Berlin
Techn. Universität Berlin

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Techn. Universität Berlin

Max Zimmer
Zuse Institut Berlin
Techn. Universität Berlin

Extending the Continuum of Six-Colorings

The Hadwiger-Nelson problem

Problem (Nelson 1950, but also Hadwiger, Erdős, Gardner, Moser, Harary, Tutte, ...)

What is the smallest number of colors sufficient for coloring the plane in such a way that no two points of the same color are a unit distance apart?

Considering the infinite graph with vertex set $\mathbb{E}^2$ and edges $\{x, y\}$ for any $x, y \in \mathbb{E}^2$ with $\|x - y\| = 1$, we are studying the **chromatic number of the plane** $\chi(\mathbb{E}^2)$.

Theorem (N.G. de Bruijn, P. Erdős 1951)

Assuming AoC *any graph is k-colorable iff every finite subgraph of it is k-colorable.*

This problem has a long and complicated history which has been well documented by Soifer over 14 pages in *The New Mathematical Coloring Book (2024)* ...

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The history of the problem

Table 3.1 Who created the chromatic number of the plane problem?

Publication	Year	Author(s)	Problem creator(s) or source named
[Gar2]	1960	Gardner	"**Leo Moser** ...writes..."
[Had4]	1961	Hadwiger (after Klee)	**Nelson**
[E61.22]	1961	Erdős	"I cannot trace the origin of this problem"
[Cro]	1967	Croft	"A long[18]-standing open problem of **Erdős**"
[Woo1]	1973	Woodall	**Gardner**
[Sim]	1976	Simmons	**Erdős**, **Harary**, and **Tutte**
[E80.38] [E81.23] [E81.26]	1980–1981	Erdős	**Hadwiger** and **Nelson**
[CFG]	1991	Croft, Falconer, and Guy	"Apparently due to **E. Nelson**"
[KW]	1991	Klee and Wagon	"Posed in 1960–61 by **M. Gardner** and **Hadwiger**"

p. 24 of The New Mathematical Coloring Book by Alexander Soifer, 2024

The history of the problem

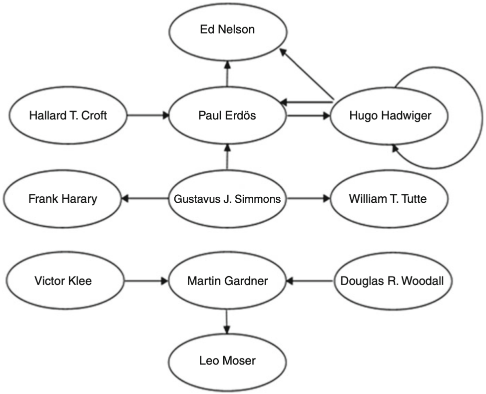

Diagram 3.1 Who created the chromatic number of the plane problem?

p. 24 of The New Mathematical Coloring Book by Alexander Soifer, 2024

The history of the problem

The results of my historical research are summarized in Diagram 3.2, where arrows show passing of the problem from one mathematician to another. In the end, Paul Erdős shares the problem with the world in numerous talks and articles.

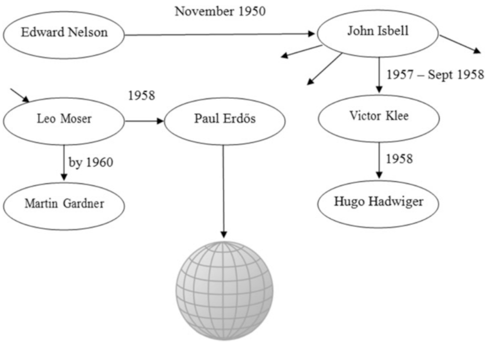

Diagram 3.2 Passing the baton of the chromatic number of the plane problem

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Lower bounds through unit distance graphs

Lower bounds are given by finding unit distance graphs of large chromatic number.

Definition

A graph $G = (V, E)$ is a *unit distance graph* if there exists an embedding $f : V \to \mathbb{E}^2$ of its vertices in the plane s.t. $\|f(u) - f(v)\| = 1$ if and only $\{u, v\} \in E$.

A triangle gives a lower bound of 3 and the Moser spindle a lower bound of 4 (1961).

Theorem (Aubrey D.N.J. de Grey, 2018)

There is a unit distance graph on 20 425 *vertices with chromatic number* 5.

Exoo and Ismailescu found a simpler construction with 627 vertices, Heule one with 553 vertices, and Jaan Parts, as part of `Polymath16`, one with 510 vertices.

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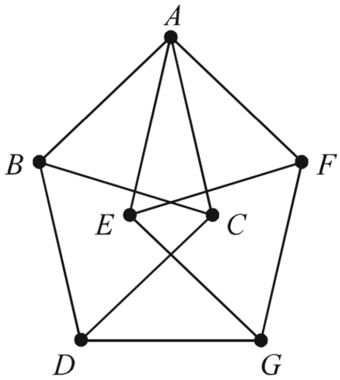

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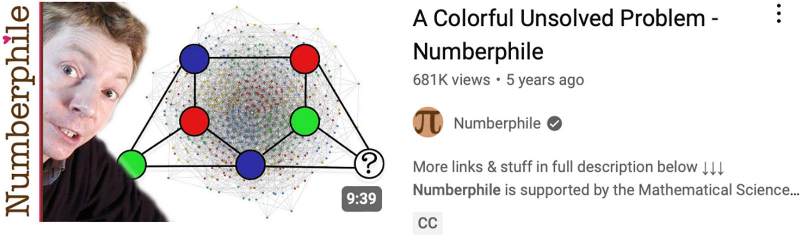

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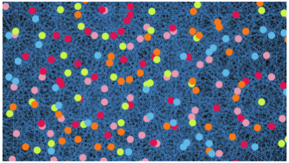

GRAPH THEORY

Decades-Old Graph Problem Yields to Amateur Mathematician

By EVELYN LAMB | APRIL 17, 2018 | 26 |

...number of vertices? The problem, now known as the Hadwiger-Nelson problem or the problem of finding the chromatic number of the plane, has piqued the interest of many mathematicians, including...

Quantamagazine

Lower bounds through unit distance graphs

Aubrey de Grey and Alexander Soifer, *Il Vicino*, January 18, 2020

Ronald L. Graham presents Aubrey D.N.J. de Grey the Prize: $1000, San Diego, September 22, 2018

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Upper bounds through colorings

Upper bounds are given by explicit colorings $g : \mathbb{E}^2 \to [c] := \{1, \ldots, c\}$, usually derived through tesselations using simple polytopal shapes, which give

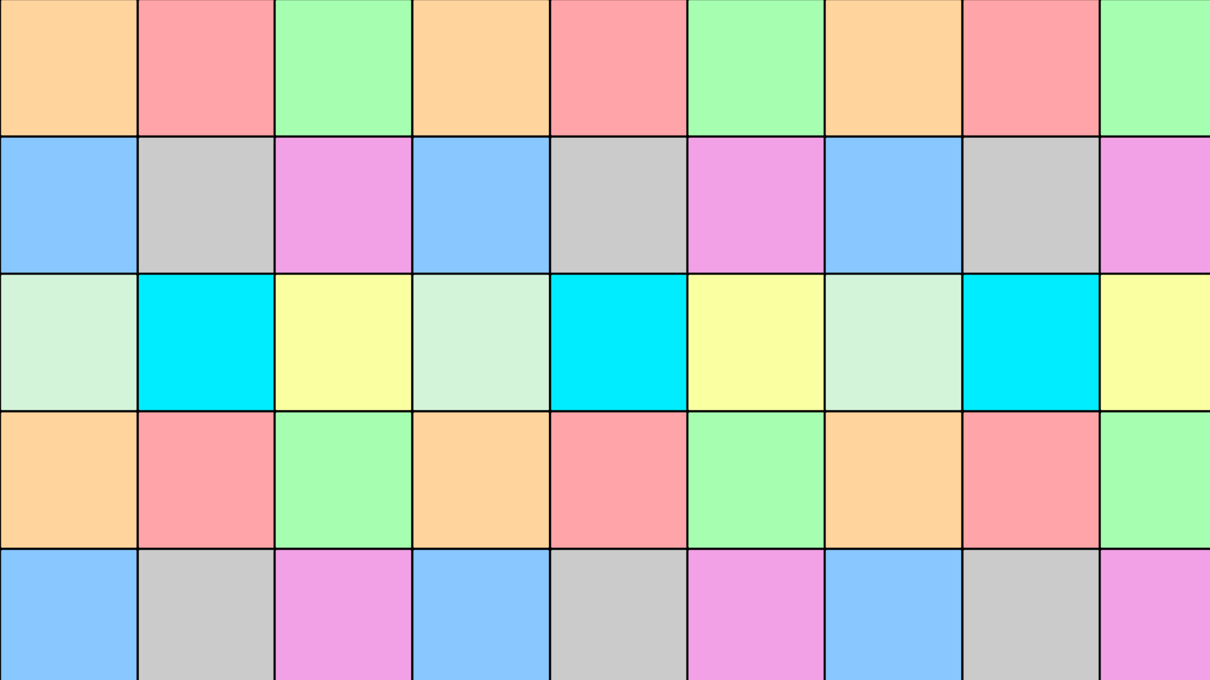
$$5 \leq \chi(\mathbb{E}^2) \leq \ldots$$

Question. Can we use computers to find colorings $g : \mathbb{E}^2 \to [c]$ so that

$$\left\{x \in \mathbb{E}^2 \mid g(x) = g(y) \text{ for any } y \in B_1(x)\right\} = \emptyset?$$

Idea. Use a parameterized and easily differentiable family $g_\theta : \mathbb{E}^2 \to \Delta_c$ and find

$$\arg\min_\theta \mathbf{E}\left[\int_{B_1(x)} g_\theta(x) \cdot g_\theta(y)\, dy \middle| x \in \mathbb{E}^2\right].$$



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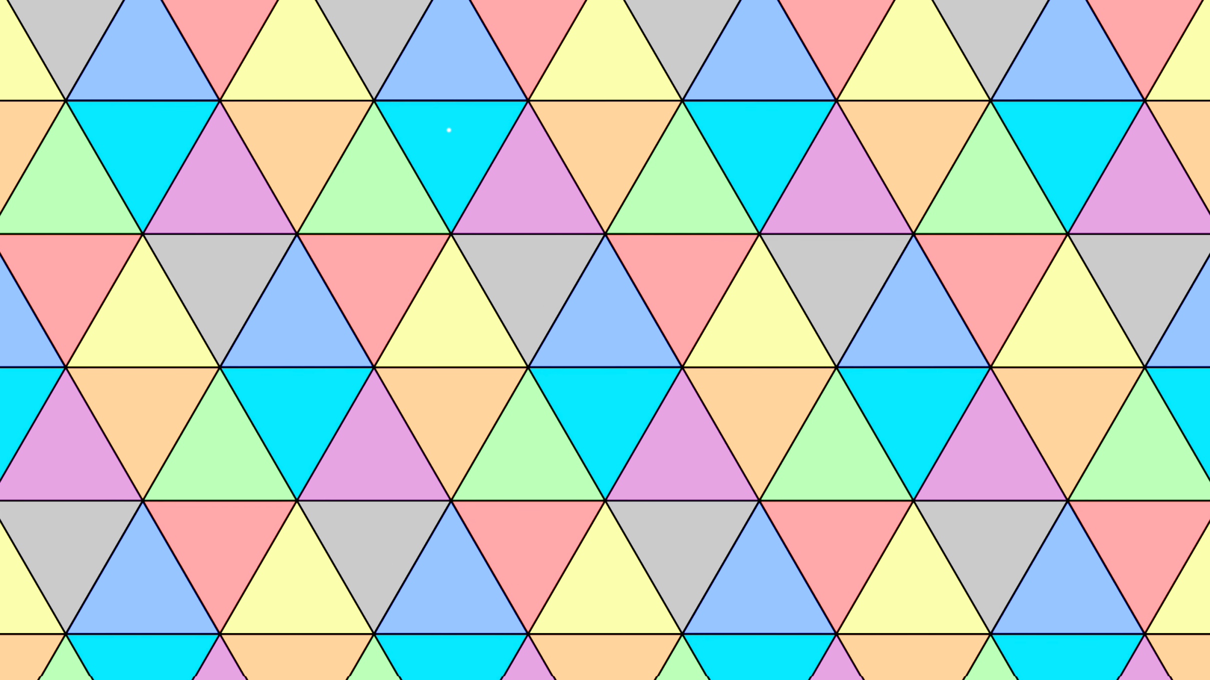

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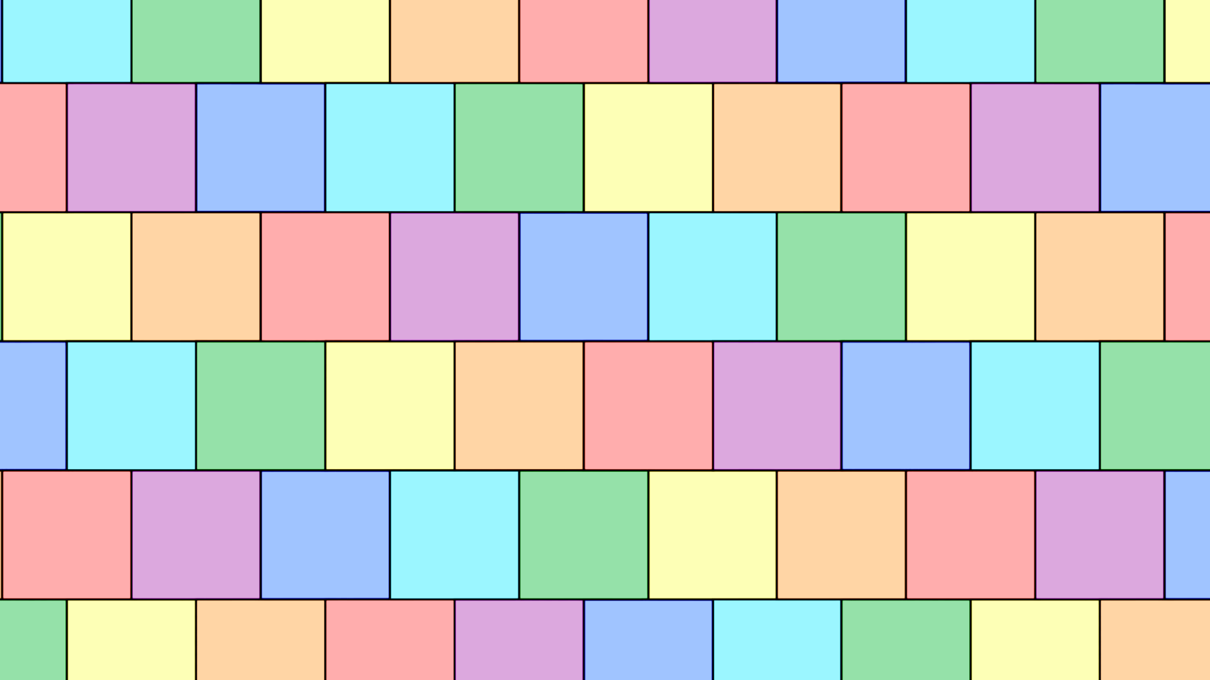

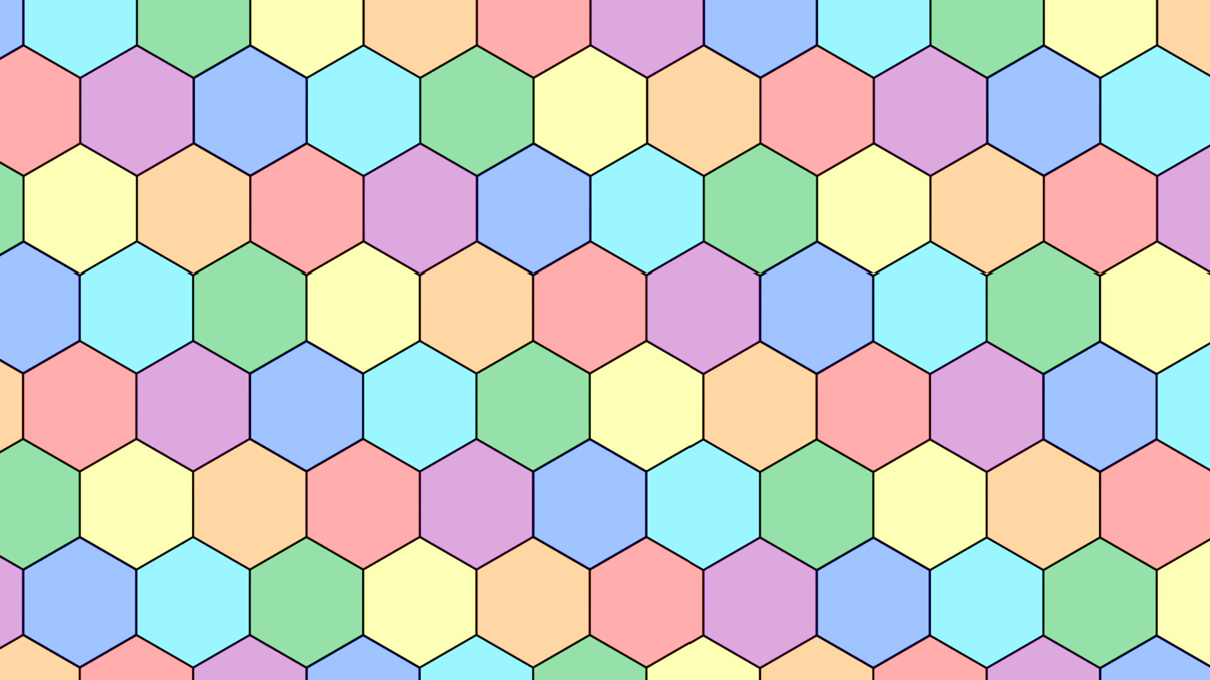

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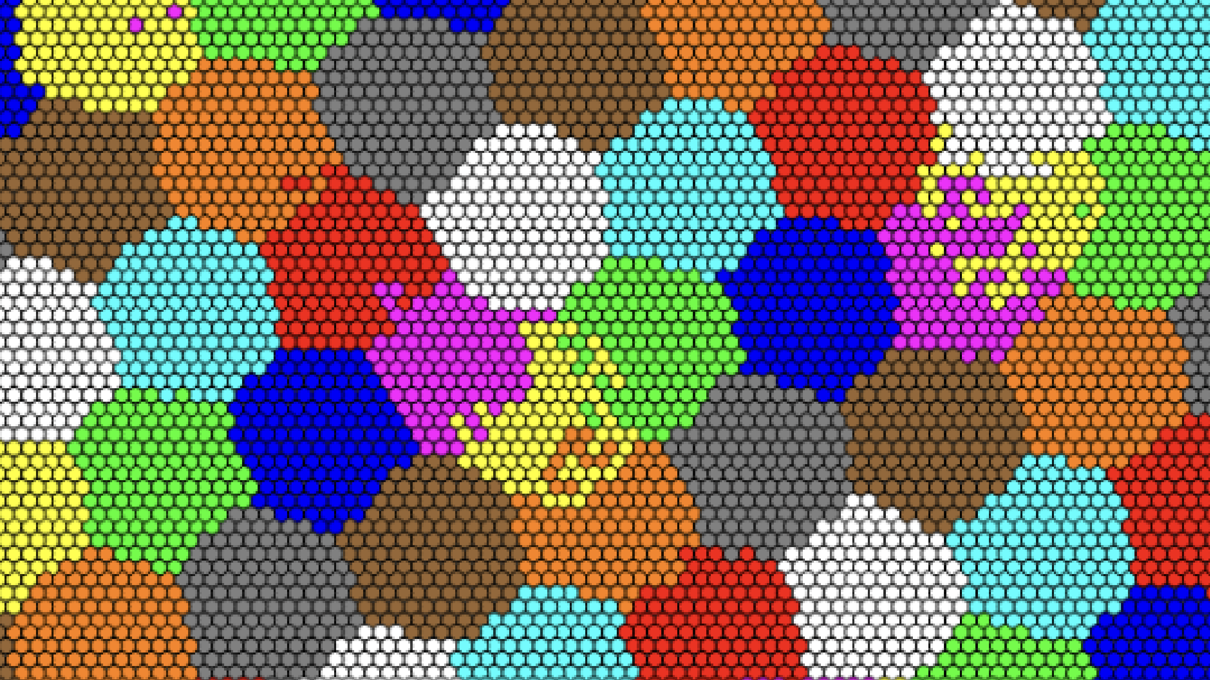

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Extending the Continuum of Six-Colorings

What are Neural Networks?

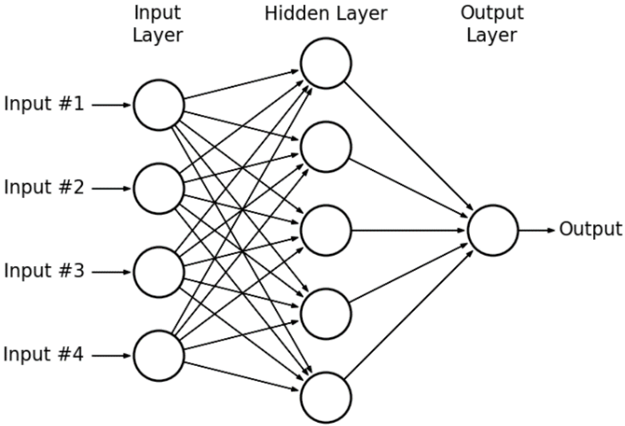

Figure: Feedforward neural network or multilayer perceptron architecture.

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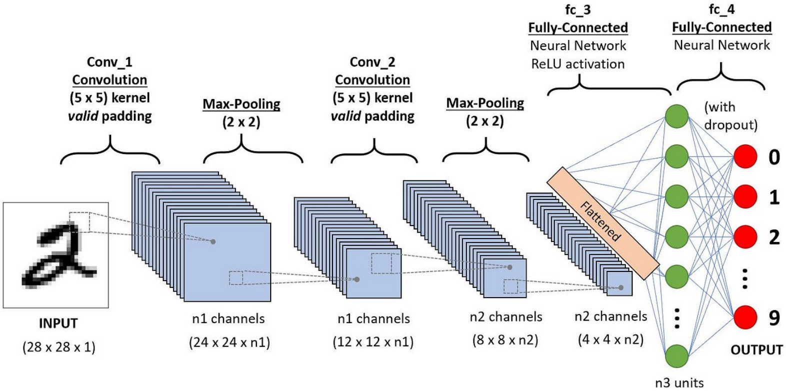

Figure: Convolutional neural network architecture.

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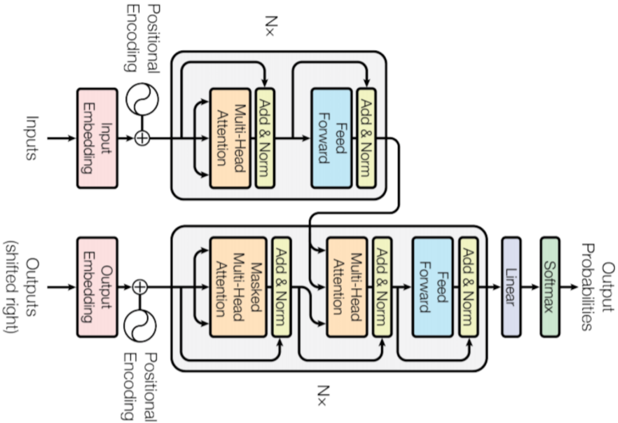
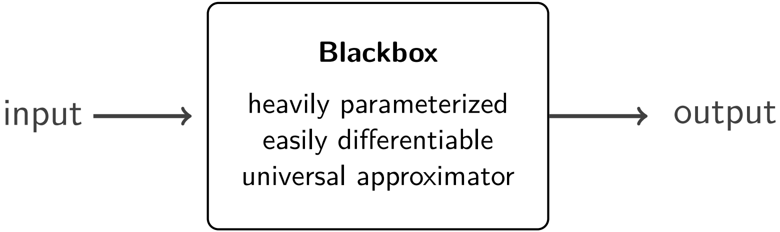


Figure: Transformer neural network architecture.

What are Neural Networks?

Just a parameterized family of functions g_θ with some convenient properties...



Theorem (Universal Approximation Theorem)

Feedforward neural networks with certain activation functions are dense (w.r.t. compact convergence) in the space of continuous functions.

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input $\longrightarrow$

Blackbox

heavily parameterized
easily differentiable
universal approximator

$\longrightarrow$ output

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Feedforward neural networks with certain activation functions are dense (w.r.t. compact convergence) in the space of continuous functions.

How do we find the correct parameters?

Idea. What if we use batch gradient descent to 'train' $g_\theta : \mathbb{E}^2 \to \Delta_6$ to minimize

$$L(\theta) = \int_{[-b,b]\times[-b,b]} \int_{B_1(x)} g_\theta(x) \cdot g_\theta(y)\, dy\, dx?$$

Algorithm. We sample points $x^{(i)} \in [-b, b] \times [-b, b]$ and $y^{(i)} \in B_1(x)$ and use that

$$\nabla_\theta L(\theta) \approx \hat{\nabla}_\theta L(\theta) := \sum_{i=1}^{m} \nabla_\theta\, g_\theta(x^{(i)}) \cdot g_\theta(y^{(i)})/m,$$

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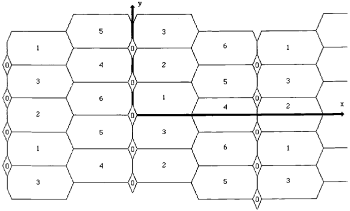
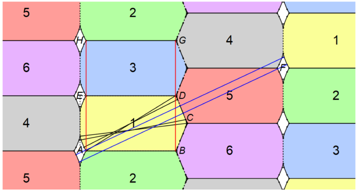
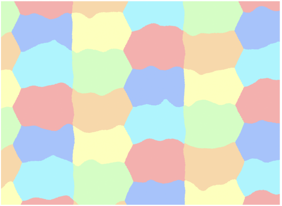


FIG. 3. A good 7-coloring of $(\mathbb{R}^2, 1)$.





Theorem (Pritikin 1995; refined by Parts 2020)

99.985% of the plane can be colored with 6 colors such that no two points of the same color are a unit distance apart.

Corollary

Any unit distance graph with chromatic number 7 must have order at least 6 993.

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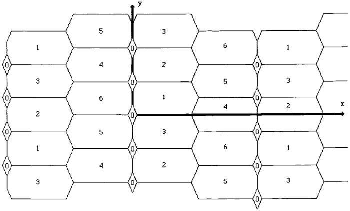

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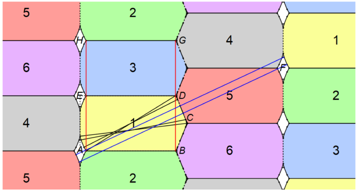

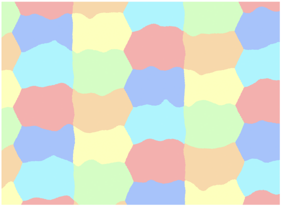

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Going off-diagonal...

A variant. A c-coloring of the plane has *coloring type* or *realizes* $(d_1, \ldots, d_c)$ if color i does not contain any points at distance d_i.

Problem (The continuum of six-colorings; Soifer in Nash and Rassias' *Open Problems in Mathematics*)

Determine the set of d for which $(1, 1, 1, 1, 1, d)$ *can be realized.*

Soifer found a coloring for $d = 1/\sqrt{5}$ in 1991. Hoffman and Soifer also found one for $d = \sqrt{2} - 1$ in 1993. Both of these are part of a family that covers any

$$0.414 \approx \sqrt{2} - 1 \leq d \leq 1/\sqrt{5} \approx 0.447.$$

Theorem (Mundinger, Pokutta, S., Zimmer 2024)

There is a coloring realizing $(1, 1, 1, 1, 1, d)$ *for any* $0.418 \leq d \leq 0.657$ *and another (family of) colorings covers any* $0.354 \leq d \leq 0.553$.

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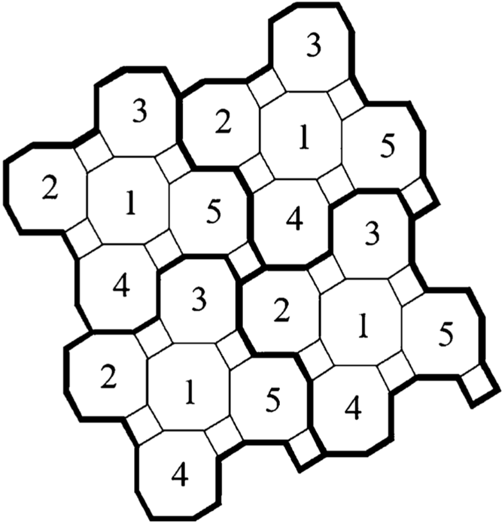
$$0.414 \approx \sqrt{2} - 1 \leq d \leq 1/\sqrt{5} \approx 0.447.$$

Theorem (Mundinger, Pokutta, S., Zimmer 2024)

There is a coloring realizing $(1, 1, 1, 1, 1, d)$ *for any* $0.418 \leq d \leq 0.657$ *and another (family of) colorings covers any* $0.354 \leq d \leq 0.553$.



3
3
2
1
5
2
1
5
4
3
4
3
2
1
5
2
1
5
4
4

Going off-diagonal...

A variant. A c-coloring of the plane has *coloring type* or *realizes* $(d_1, \ldots, d_c)$ if color i does not contain any points at distance d_i.

Problem (The continuum of six-colorings; Soifer in Nash and Rassias' *Open Problems in Mathematics*)

Determine the set of d for which $(1, 1, 1, 1, 1, d)$ *can be realized.*

Soifer found a coloring for $d = 1/\sqrt{5}$ in 1991. Hoffman and Soifer also found one for $d = \sqrt{2} - 1$ in 1993. Both of these are part of a family that covers any

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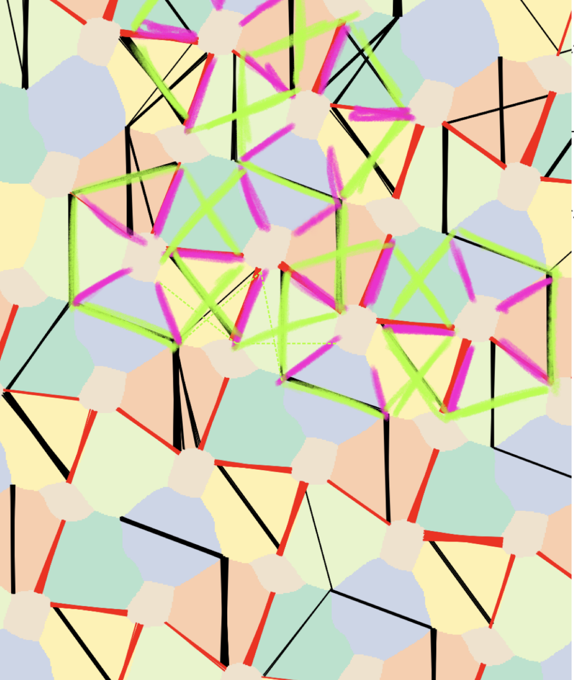
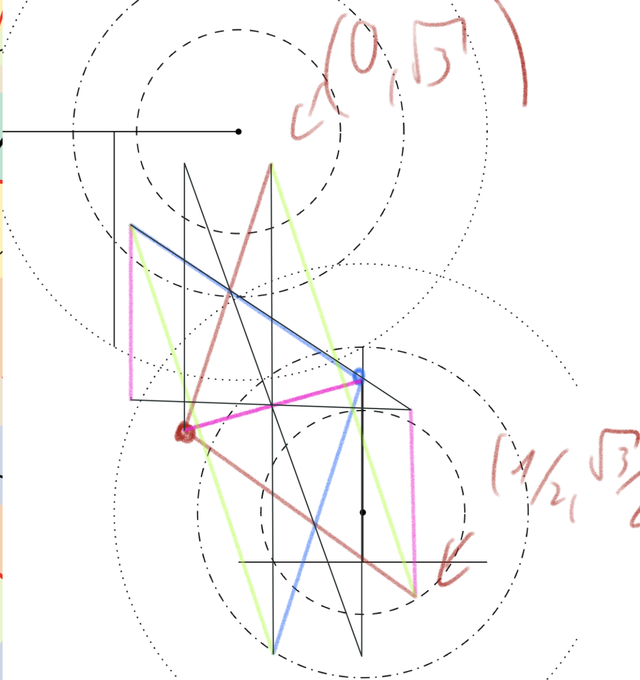
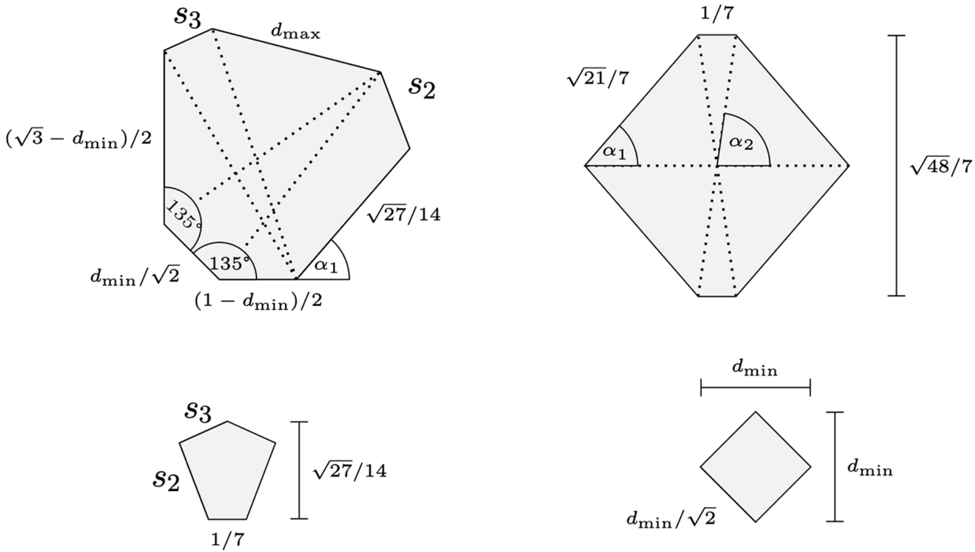
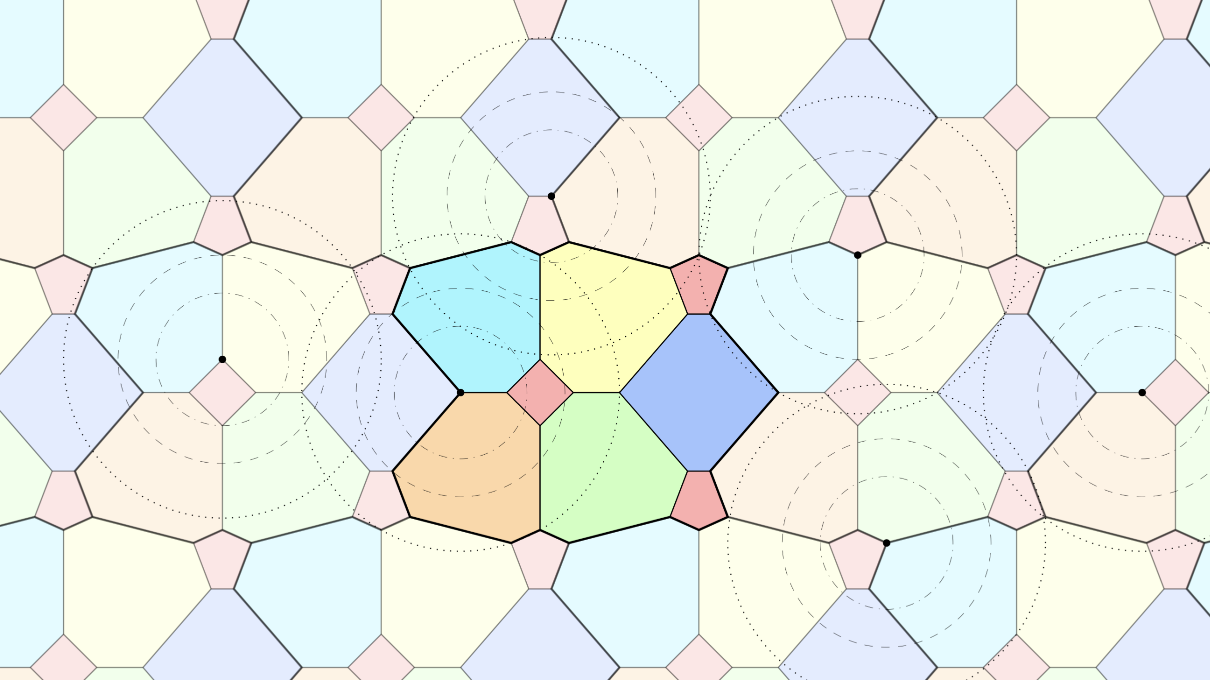
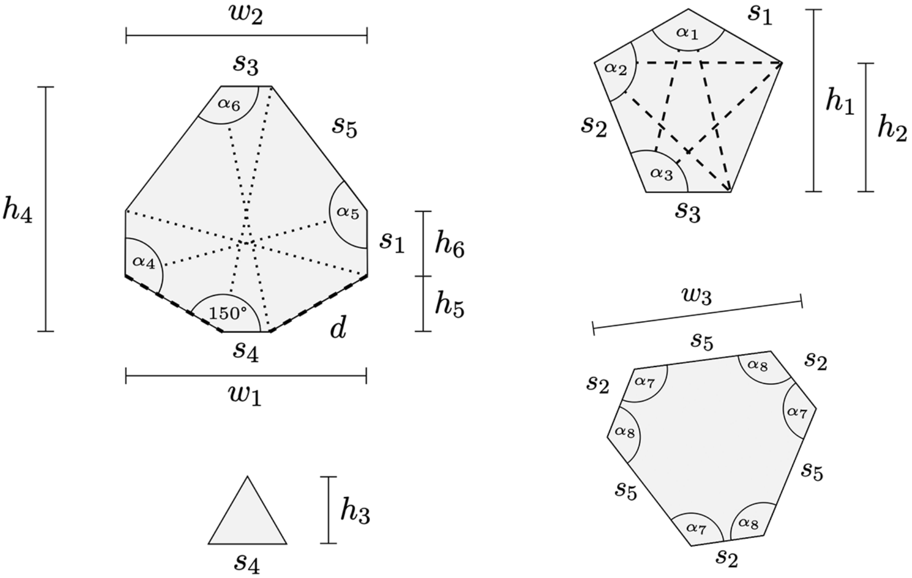
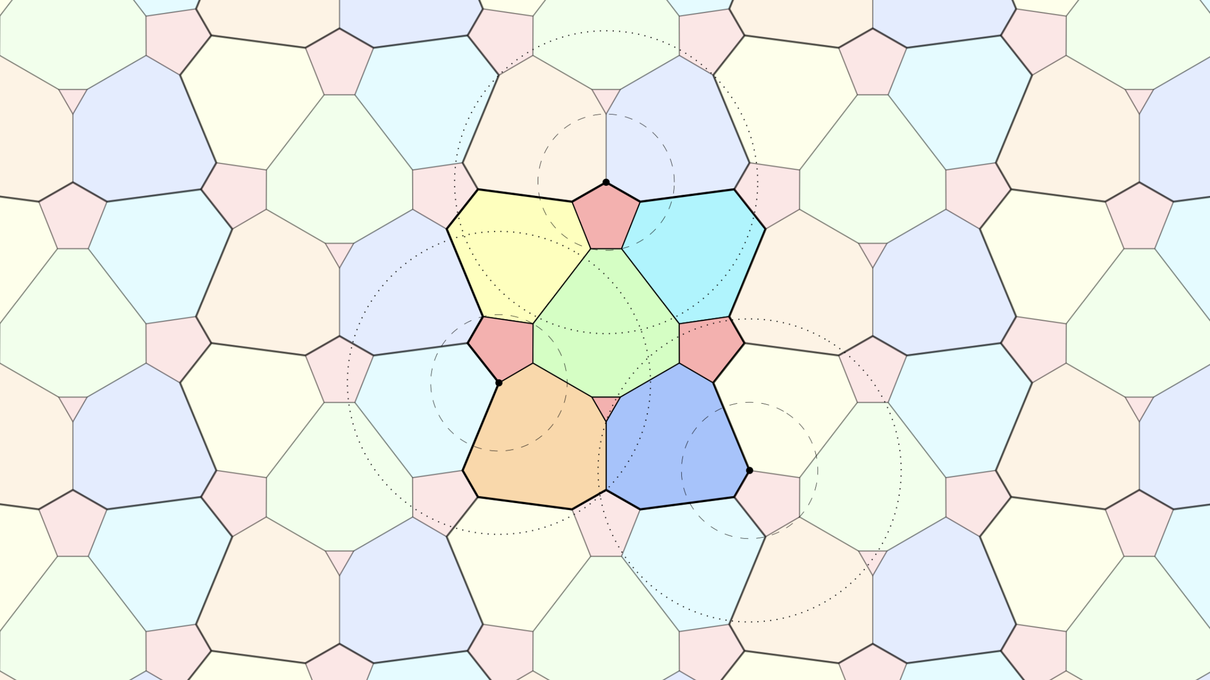



(0, √3)
(1/2, √3/


s_3
$d_{\max}$
s_2
$(\sqrt{3} - d_{\min})/2$
135°
135°
$\sqrt{27}/14$
α_1
$d_{\min}/\sqrt{2}$
$(1 - d_{\min})/2$
$1/7$
$\sqrt{21}/7$
α_1
α_2
$\sqrt{48}/7$
s_3
s_2
$\sqrt{27}/14$
$1/7$
$d_{\min}$
$d_{\min}$
$d_{\min}/\sqrt{2}$




w_2
s_3
α_6
s_5
h_4
α_5
s_1
h_6
α_4
h_5
150°
d
s_4
w_1
h_3
s_4
s_1
α_1
α_2
h_1
h_2
s_2
α_3
s_3
w_3
s_5
s_2
α_8
s_2
α_7
α_7
α_8
s_5
s_5
α_7
α_8
s_2



Is this optimal?

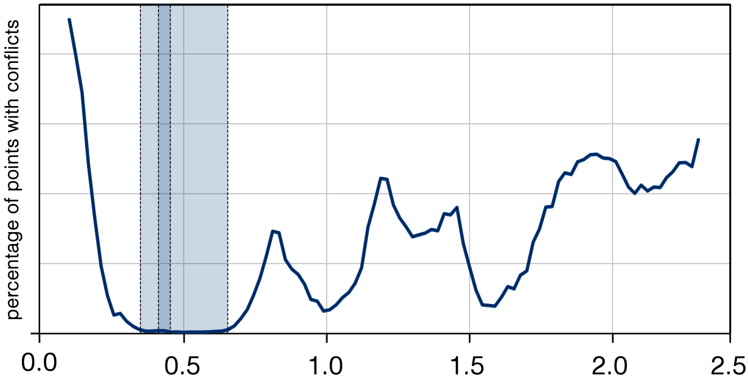


Figure: Numerical results showing the percentage of points with some conflict for a given forbidden distance *d* in the sixth color found over several minimized over several runs.

Open problems and final remarks

The underlying optimization approach is very flexible:

- Can we formalize these colorings as **Voronoi diagrams**?
- Can we improve the upper bound of the **chromatic number of** $\mathbb{E}^3$ from 15 to 14?
- Can we apply the same ideas to generate **graphons and other limit structures**?
- Can we use **adversarial networks** when the objectiv is non-differentiable?

Full description of the two colorings is available at `arxiv.org/abs/2404.05509`.

Thank you for your attention!