Flag algebras in additive combinatorics

DOxML 2023 at GRIPS

Juanjo Rué Perna and **Christoph Spiegel**

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Flag Algebras in Additive Combinatorics

1. Additive Combinatorics — 4 slides
2. Constructive upper bounds through blow-ups — 3 slides
3. Double counting on steroids — 5 slides
4. Conclusion — 1 slide

Additive Combinatorics 101

Let G be a finite (abelian) group of order N or an interval $[N] \stackrel{\text{def}}{=} \{1, \ldots, N\} \subset \mathbb{N}$.

What can we say about the (linear) *structure* of subsets $S \subseteq G$ or colorings $\gamma : G \to [c]$?

Global properties. What can we say about the relation of $|S|$ and Minkowski sumsets like $|S + S|$? *Cauchy–Davenport, Vosper, Plünnecke–Ruzsa, Freiman-Ruzsa, ...*

Local properties. Do S or G contain (monochromatic) k-term arithmetic progressions $(x, x + d, x + 2\,d, \ldots, x + (k-1)\,d)$, Schur triples $(x + y = z)$, repeated sums $(x + y = u + v)$? *Schur, van der Waerden, Rado, Szémeredi, arithmetic regularity, Green-Tao, ...*

We will focus on the latter, in particular on the Rado Multiplicity Problem!

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The Rado Multiplicity problem

Given a **coloring** $\gamma : G \to [c]$ and linear map $L : G^m \to G^n$, we are interested in

$$\mathcal{S}_L(\gamma, G) \overset{\text{def}}{=} \{\mathbf{s} \in G^m : L(\mathbf{s}) = \mathbf{0}, s_i \neq s_j \text{ for } i \neq j, \mathbf{s} \in \gamma^{-1}(\{i\})^m \text{ for some } i\}. \quad (1)$$

Let $\Gamma_c(G)$ denotes all c-colorings of G. The **Rado Multiplicity Problem** is

$$m_{q,c}(L, G) \overset{\text{def}}{=} \min_{\gamma \in \Gamma_c(G)} |\mathcal{S}_L(\gamma, G)| \,/\, |\mathcal{S}_L(G)|$$

and in particular $m_{q,c}(L) \overset{\text{def}}{=} \limsup_{n \to \infty} m_{q,c}(L, G_n)$ when $G_n = [n], \mathbb{Z}_n, \mathbb{F}_q^n$.

Rado (1933) tells us that $\mathcal{S}_L(\gamma, G_n) \neq \emptyset$ if L satisfies *column condition* and n is large, which can also be shown to imply $0 < m_{q,c}(L) < 1$.

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History of the problem

- Graham et al. (1996) gave lower bound for **Schur triples** in 2-colorings of $[n]$, later independently resolved by Robertson and Zeilberger / Schoen / Datskovsky.
- Cameron et al. (2007) showed that the nr. of solutions for linear equations with **an odd nr. of variables** only depends on cardinalities of the two color classes.
- Parrilo, Robertson and Saracino (2008) established bounds for the minimum number of **monochromatic** 3-**APs** in 2-colorings of $[n]$ (not 2-common in $\mathbb{N}$).
- For $r = 1$ and m even, Saad and Wolf (2017) showed that any 'pair-partitionable' L gets its multiplicity from uniform random colorings of $\mathbb{F}_q^n$. Fox, Pham, and Zhao (2021) showed that this is necessary and Versteegen (2023) further generalized it.
- Kamčev et al. (2021) characterized some L in $\mathbb{F}_q^n$ with $r > 1$ where the multiplicity does *not* come from random constructions.
- Král et al. (2022) characterized L where the mulitplicty comes from random constructions for $q = 2$, $r = 2$, m odd.

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Our results

We are interested in particular L and $\mathbb{F}_q^n$.

Theorem (Rué and S., 2023)

We have $1/10 < m_{q=5,c=2}(L_{4\text{-}AP}) \leq 0.1\overline{031746}$.

Saad and Wolf (2017) previously established an u.b. of 0.1247 with no no-trivial l.b. known.

Proposition (Rué and S., 2023)

We have $m_{q=3,c=3}(L_{3\text{-}AP}) = 1/27$.

Similar to Cummings et al. (2013) extending a result of Goodman (1959) about triangles.

Both upper and lower bounds are computational in nature:

Upper bounds through blow-up constructions of particular finite colorings. `Discrete and Comb. Optimization`

Lower bounds by extending Razborov's Flag Algebra framework. `Conic Optimization, Sum-of-Squares, and Semidefinite Programming`

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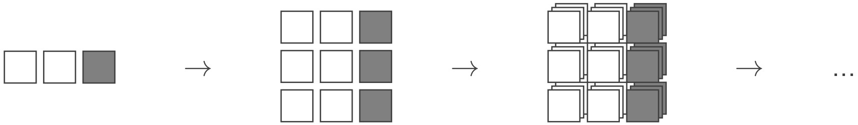
How to blow up colorings

What do we need for upper bounds? Sequences of colorings of increasing size...

How can we turn this into a finite problem? By considering *blowups*:



Relevant in other contexts, e.g., Turán and Ramsey theory, capset problem, Sunflower conjecture, Turán's $(3, 4)$-conjecture, the Shannon Capacity of odd cycles...

Lemma

The limit of the density of monochromatic structures in the blow-up sequence is the non-injective density of monochromatic structures in the base coloring.

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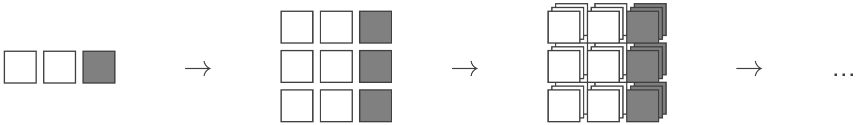

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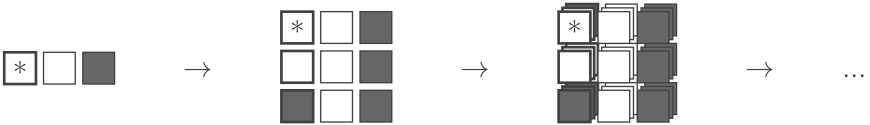

You can find constructions to blow up using your favorite Discrete Optimization technique:

isomorphism-free generation, SAT-solver, Integer Linear Programming, Bounded Tree Searches, Search Heuristics (Simmulated Annealing, Tabu Search, Genetic algorithms), even Machine Learning, ...

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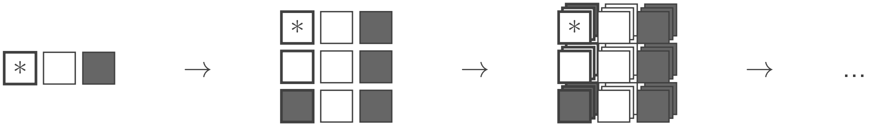

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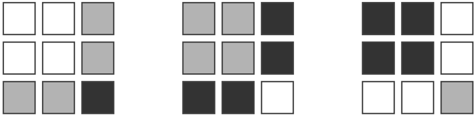
Proofs of the upper bounds

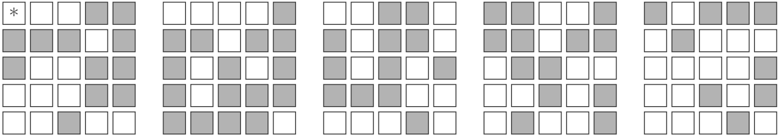
Upper bound of the Theorem

$m_{3,3}(L_{4\text{-AP}}) \leq 1/27$ follows from the blow-up of this 3-coloring of $\mathbb{F}_3^3$:



Upper bound of the Proposition

$m_{5,2}(L_{4\text{-AP}}) \leq 13/126$ follows from the iterated blow-up of this 2-coloring of $\mathbb{F}_5^3$:



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An improvement on a trivial lower bound

The parameter $s_L(\gamma) \stackrel{\text{def}}{=} |\mathcal{S}_L(\gamma)| \,/\, |\mathcal{S}_L(\mathbb{F}_q^n)|$ satisfies the averaging equality

$$s_L(\gamma) = \sum_{\delta \in \Gamma(k)} p(\delta,\gamma)\, s_L(\delta) + o(1) = \mathbb{E}^{(\gamma)}_{\delta \in \Gamma(k)} s_L(\delta) + o(1) \tag{2}$$

once k is large enough. This implies an immediate trivial lower bound of

$$m_{q,c}(L) \geq \min_{\delta \in \Gamma(k)} s_L(\delta). \tag{3}$$

If we *magically* found some coefficients a_δ satisfying $\mathbb{E}^{(\gamma)}_{\delta \in \Gamma(k)} a_\delta = o(1)$, we would get

$$m_{q,c}(L) \geq \min_{\delta \in \Gamma(k)} s_L(\delta) - a_\delta. \tag{4}$$

But how would we find such a_δ? **Flag Algebras and Semidefinite Programming!**

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SOS please someone help me

Definition

The *flag algebra* $\mathcal{A}$ is given by considering linear combinations of colorings, factoring out relations given by the averaging equality and defining an appropriate product.

The *semantic cone* $\mathcal{S} = \{f \in \mathcal{A} : \phi(f) \geq 0 \text{ for all } \phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})\}$ captures those **algebraic expressions corresponding to density expressions that are 'true'**.

There exists an element $C_L \in \mathcal{A}$ capturing the behavior of s_L, so we can establish a lower bound by establishing an SOS expression

$$C_L - \lambda - \sum_{i=1}^{k} f_i^2 \in \mathcal{S}. \tag{5}$$

The $p(\sum_{i=1}^{k} f_i^2, \delta)$ correspond to the a_δ on the previous slide!

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Challenges

1. Need an appropriate notion of density, isomorphism, and ‘partially fixed coloring’ both to (i) handle invariance and non-invariance and (ii) define different algebras.

2. Solutions as defined previously do *not* satisfy an exact averaging equality.
   $\rightarrow$ Introduce *fully dimensional solutions*, which asympt. make up all solutions.

3. Need to adequately solve isomorphisms problem from a practical perspective.
   $\rightarrow$ Represent structure as graph and use `nauty`.

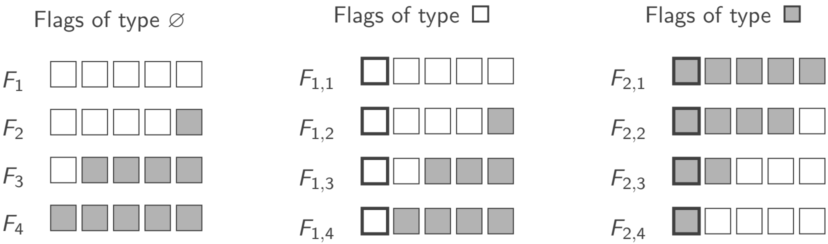
4. (Almost) all SDP solvers work numerically, but we need algebraic expressions.
   $\rightarrow$ Refine solution an using exact LP solver like `SoPlex`.

Lower bound of the Proposition

$m_{5,2}(L_{4\text{-AP}}) > 1/10$ follows by verifying that over all 3324 2-colorings of $\mathbb{F}_5^2$ we have

$$F_1 + F_4 + (F_2 + F_3)/5 - 1/10 \geq \sum_{i=1}^{2} \Big(9/10 \cdot \big[\!\big[(F_{i,1} + (5\,F_{i,2} - 5\,F_{i,3} - 10\,F_{i,4})/27)^2\big]\!\big]_{-1}$$
$$\ldots + 61/162 \cdot \big[\!\big[((F_{i,3} - F_{i,2})/2 + F_{i,4})^2\big]\!\big]_{-1}\Big),$$

and by noting that $F_{1,1} + F_{2,1} > 0$. Here the relevant flags F_i and $F_{i,j}$ are

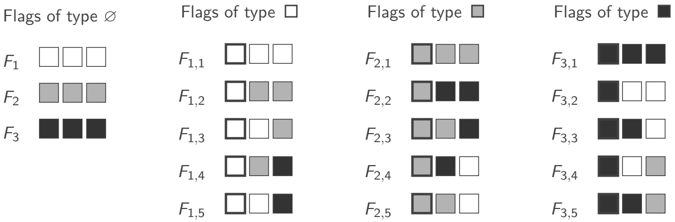


Lower bound of the Theorem

$m_{3,3}(L_{3\text{-AP}}) \geq 1/27$ follows by verifying that over all all 140 3-colorings of $\mathbb{F}_3^2$ we have

$$\begin{aligned} F_i - 1/27 \geq 26/27 \cdot & [\![(F_{i,1} - 99/182\, F_{i,2} + 75/208\, F_{i,3} - 11/28\, F_{i,4} - 3/26\, F_{i,5})^2]\!]_{-1} \\ & \ldots + 1685/1911 \cdot [\![(F_{i,2} - 231/26960\, F_{i,3} + 1703/6740\, F_{i,4} - 1869/3370\, F_{i,5})^2]\!]_{-1} \\ & \ldots + 71779/431360 \cdot [\![(F_{i,3} - 358196/502453\, F_{i,4} - 412904/502453\, F_{i,5})^2]\!]_{-1} \\ & \ldots + 5431408/10551513 \cdot [\![(F_{i,4} - 1/4\, F_{i,5})^2]\!]_{-1} \end{aligned}$$

for any $i \in \{1, 2, 3\}$. Here the relevant flags F_i and $F_{i,j}$ are



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Open problems and final remarks

- Often one can extract stability results from Flag Algebra certificates.
- Steep computational hurdle: underlying structures grow exponentially (instead of quadratically for graphs or cubic for 3-uniform hypergraphs)
- No neat notion of subspaces makes generalizing to other groups difficult.

Code is available at `github.com/FordUniver/rs_radomult_23`

Thank you for your attention!