Generalized Positional van der Waerden Games

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Maker-Breaker Positional Games

Definition (Maker-Breaker Games)

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Theorem (Erdős-Selfridge '73, Beck '82)

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There is also a much weaker, rarely used Maker's criterion due to Beck.

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Example (van der Waerden Game – Beck '81)

Van der Waerden games are the positional games played on the board $[n] = \{1, \ldots, n\}$ with all k-AP as winning sets.

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What about the biased version?

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What about more general additive structures?

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For $\emptyset \subseteq Q \subseteq [m]$, let A^Q denote the matrix keeping only columns indexed by Q and let $r_Q = \mathrm{rk}(A) - \mathrm{rk}(A^{\overline{Q}})$.

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Definition (Maximum 1-density)

For $\emptyset \subseteq Q \subseteq [m]$, let A^Q denote the matrix keeping only columns indexed by Q and let $r_Q = \text{rk}(A) - \text{rk}(A^{\overline{Q}})$. The *maximum* 1-*density* of an abundant matrix $A \in \mathbb{Z}^{r \times m}$ is defined as

$$m_1(A) = \max_{\substack{Q \subseteq [m] \\ 2 \leq |Q|}} \frac{|Q| - 1}{|Q| - r_Q - 1}. \tag{1}$$

Example (Schur triple)

The matrix associated with Schur triple is given by

$$A_{\text{Schur}} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \in \mathbb{Z}^{1\times 3}. \tag{2}$$

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Example (Arithmetic Progressions)

The matrix associated with a k-term arithmetic progression is given by

$$A_{k\text{-AP}} = \begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \cdots & & & \\ & & & & 1 & -2 & 1 \end{pmatrix} \in \mathbb{Z}^{(k-2)\times k}. \tag{3}$$

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$\begin{pmatrix} 1 & 1 & -2 \end{pmatrix}$ *is density regular and* $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ *is partition regular. However,* $\begin{pmatrix} 1 & 1 & -3 \end{pmatrix}$ *is abundant but neither density nor partition regular.*

van der Waerden Games

Definition

Given any matrix $A \in \mathbb{Z}^{r \times m}$ let the corresponding *generalized van der Waerden Game* be the Maker-Breaker positional game with $[n]$ as the board and $\{\mathbf{x} \in [n]^m : A \cdot \mathbf{x}^T = \mathbf{0}^T,\ x_i \neq x_j\}$ as the winning sets.

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Theorem (Kusch, Rué, S. and Szabó '17)

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Corollary

For van der Waerden games the threshold is $\Theta\left(n^{1/(k-1)}\right)$ *for* $k \geq 3$.

There are also a results allowing some repeated entries and results dealing with the inhomogeneous case.

Proof Outline

Bednarska and Łuczak ′00 studied the bias threshold of the Maker-Breaker game consisting of all copies of a given *small* graph G in K_n.

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In our paper we extend the ideas behind their proof to obtain general Maker and Breaker Win Criteria and apply them to the van der Waerden games. These General criteria also allow one to generalize the result of Bednarska and Łuczak to hypergraphs of higher uniformity.

Here I will use the stronger ingredient of a probabilistic Ramsey statement for Maker’s part and give an outline of the proof for Breaker’s strategy.

Maker's Strategy: *playing randomly*

Theorem (Schacht; Conlon and Gowers '10)

For all positive and **density regular** $A \in \mathbb{Z}^{r\times m}$ *and* $\varepsilon > \mathbf{0}$ *there exist* c, C*:*

$$\lim_{n\to\infty} \mathbb{P}\left([n]_p \to_\varepsilon A\right) = \begin{cases} 0 & \text{if } p(n) \le c\, n^{-1/m_1(A)}, \\ 1 & \text{if } p(n) \ge C\, n^{-1/m_1(A)}. \end{cases}$$

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Theorem (Hancock, Staden and Treglown '17+; S. '17+)

For every positive and **abundant** *matrix* $A \in \mathbb{Z}^{r\times m}$ *and* $\boldsymbol{\varepsilon} > \boldsymbol{\pi}(\boldsymbol{A})$ *there exist constants* $c(A, \varepsilon), C(A, \varepsilon) > 0$ *such that*

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Proof.

Let an arbitrary strategy for Breaker be fixed.

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5. By the previous result, Maker's random response succeeds a.a.s. so that there must exist a deterministic winning strategy. □

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$A[Q_1]$ is positive and abundant. Furthermore, blocking solutions to $A[Q_1]$ also blocks solutions to A:

Lemma

Let $T \subset \mathbb{N}$ and $Q_1 \subseteq [m]$ as above. If there does not exist a solution to $A[Q_1] \cdot \boldsymbol{x}^T = \mathbf{0}^T$ in T then there also does not exist a solution to $A \cdot \boldsymbol{x}^T = \mathbf{0}^T$.

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Remark (Strategy Splitting)

If Breaker has a winning strategy in $\mathcal{H}_1$ and $\mathcal{H}_2$ with a bias of q_1 and q_2 respectively, then he has a winning strategy in $\mathcal{H}_1 \cup \mathcal{H}_2$ with a bias of $q_1 + q_2$.

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For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ *Breaker wins the associated van der Waerden game with a bias of* $q \gg n^{1/m_1(A)}$.

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To get to the correct threshold, one combines a strategy as above aimed at structures **intersecting in at least** 2 **points** *with another application of Erdős-Selfridge aimed at structures* **intersecting in exactly** 1 **point**.

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For all positive and abundant matrices $A \in \mathbb{Z}^{r\times m}$ there exists a constant $c = c(A)$ such that for $\varepsilon > 0$ and n large enough Breaker has a winning strategy with a bias of $q > (c+\varepsilon)\, n^{1/m_1(A)}$ and Maker has a winning strategy if $q < (c-\varepsilon)\, n^{1/m_1(A)}$.

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Q2. Can one formulate an explicit winning strategy for Maker?

Thank you for your attention!