The Rado Multiplicity Problem in $\mathbb{F}_q^n$

Eurocomb 2023 at Charles University

Juanjo Rué Perna and Christoph Spiegel

28th of August 2023
The Rado Multiplicity Problem in $\mathbb{F}_q^n$

1. The Rado Multiplicity Problem 3 slides

2. Constructive Upper Bounds through Blow-ups 2 slides

3. Lower Bounds through Flag Algebras 4 slides

4. Concluding Remarks and Open Problems 1 slide
1. The Rado Multiplicity Problem

The Rado Multiplicity problem

Given a coloring $\gamma : \mathbb{F}_q^n \to [c]$ and linear map $L$, we are interested in

$$S_L(\gamma) \overset{\text{def}}{=} \{ s \in (\mathbb{F}_q^n)^m : L(s) = 0, s_i \neq s_j \text{ for } i \neq j, s \in \gamma^{-1}(\{i\})^m \text{ for some } i \}. \quad (1)$$

Rado (1933) tells us that $S_L(\gamma) \neq \emptyset$ for large enough $n$ if $L$ satisfies column condition.

The Rado Multiplicity Problem is concerned with determining

$$m_{q,c}(L) \overset{\text{def}}{=} \lim_{n \to \infty} \min_{\gamma \in \Gamma(n)} |S_L(\gamma)| / |S_L(\mathbb{F}_q^n)|.$$

Limit exists by monotonicity and $0 < m_{q,c}(L) \leq 1$ if $L$ is partition regular. $L$ is $c$-common if $m_{q,c}(L) = c^{1-m}$ (the value attained in a uniform random coloring).

This reminds us of an old question of Erdős in graph theory ...
The Rado Multiplicity Problem

Given a coloring $\gamma : \mathbb{F}_q^n \to [c]$ and linear map $L$, we are interested in

$$S_L(\gamma) \overset{\text{def}}{=} \{ s \in (\mathbb{F}_q^n)^m : L(s) = 0, s_i \neq s_j \text{ for } i \neq j, s \in \gamma^{-1}(\{i\})^m \text{ for some } i \}. \quad (1)$$

Rado (1933) tells us that $S_L(\gamma) \neq \emptyset$ for large enough $n$ if $L$ satisfies column condition.

The Rado Multiplicity Problem is concerned with determining

$$m_{q,c}(L) \overset{\text{def}}{=} \lim_{n \to \infty} \min_{\gamma \in \Gamma(n)} \frac{|S_L(\gamma)|}{|S_L(\mathbb{F}_q^n)|}.$$ 

Limit exists by monotonicity and $0 < m_{q,c}(L) \leq 1$ if $L$ is partition regular. $L$ is $c$-common if $m_{q,c}(L) = c^{1-m}$ (the value attained in a uniform random coloring).

This reminds us of an old question of Erdős in graph theory ...
The Rado Multiplicity Problem

The Rado Multiplicity problem

Given a coloring $\gamma : \mathbb{F}_q^n \to [c]$ and linear map $L$, we are interested in

$$S_L(\gamma) \overset{\text{def}}{=} \{ s \in (\mathbb{F}_q^n)^m : L(s) = 0, s_i \neq s_j \text{ for } i \neq j, s \in \gamma^{-1}\{\{i\}\}^m \text{ for some } i \}. \quad (1)$$

Rado (1933) tells us that $S_L(\gamma) \neq \emptyset$ for large enough $n$ if $L$ satisfies column condition.

The Rado Multiplicity Problem is concerned with determining

$$m_{q,c}(L) \overset{\text{def}}{=} \lim_{n \to \infty} \min_{\gamma \in \Gamma(n)} |S_L(\gamma)| / |S_L(\mathbb{F}_q^n)|.$$

Limit exists by monotonicity and $0 < m_{q,c}(L) \leq 1$ if $L$ is partition regular. $L$ is $c$-common if $m_{q,c}(L) = c^{1-m}$ (the value attained in a uniform random coloring).

This reminds us of an old question of Erdős in graph theory ...
1. The Rado Multiplicity Problem

**History of the problem**

- Graham et al. (1996) gave lower bound for Schur triples in 2-colorings of \([n]\), later independently resolved by Robertson and Zeilberger / Schoen / Datskovsky.
- Cameron et al. (2007) showed that the nr. of solutions for linear equations with an odd nr. of variables only depends on cardinalities of the two color classes.
- Parrilo, Robertson and Saracino (2008) established bounds for the minimum number of monochromatic 3-APs in 2-colorings of \([n]\) (not 2-common in \(\mathbb{N}\)).
- For \(r = 1\) and \(m\) even, Saad and Wolf (2017) showed that any ‘pair-partitionable’ \(L\) is 2-common in \(\mathbb{F}_q^n\). Fox, Pham, and Zhao (2021) showed that this is necessary and Versteegen further generalized their result.
- Kamčev et al. (2021) characterized some non-common \(L\) in \(\mathbb{F}_q^n\) with \(r > 1\).
- Král et al. (2022) characterized 2-common \(L\) for \(q = 2, r = 2, m\) odd.
1. The Rado Multiplicity Problem

History of the problem

- Graham et al. (1996) gave lower bound for **Schur triples** in 2-colorings of $[n]$, later independently resolved by Robertson and Zeilberger / Schoen / Datskovsky.

- Cameron et al. (2007) showed that the nr. of solutions for linear equations with **an odd nr. of variables** only depends on cardinalities of the two color classes.

- Parrilo, Robertson and Saracino (2008) established bounds for the minimum number of **monochromatic 3-APs** in 2-colorings of $[n]$ (not 2-common in $\mathbb{N}$).

- For $r = 1$ and $m$ even, Saad and Wolf (2017) showed that any ‘pair-partitionable’ $L$ is 2-common in $\mathbb{F}_q^n$. Fox, Pham, and Zhao (2021) showed that this is necessary and Versteegen further generalized their result.

- Kamčev et al. (2021) characterized some non-common $L$ in $\mathbb{F}_q^n$ with $r > 1$.

- Král et al. (2022) characterized 2-common $L$ for $q = 2$, $r = 2$, $m$ odd.
1. The Rado Multiplicity Problem

**Our results**

We are interested in particular $L$ and $F^n_q$.

**Theorem (Rué and S., 2023)**

*We have $1/10 < m_{q=5,c=2}(L_{4\cdot AP}) \leq 0.1031746$.***

Saad and Wolf (2017) previously established an u.b. of 0.1247 with no non-trivial l.b. known.

**Proposition (Rué and S., 2023)**

*We have $m_{q=3,c=3}(L_{3\cdot AP}) = 1/27$.***

Similar to Cummings et al. (2013) extending a result of Goodman (1959) about triangles.

**Upper bounds** obtained through blow-up constructions of particular finite colorings.

**Lower bounds** obtained by extending Razborov’s Flag Algebra framework.
1. The Rado Multiplicity Problem

Our results

We are interested in particular $L$ and $F^n_q$.

**Theorem (Rué and S., 2023)**

*We have $1/10 < m_{q=5, c=2}(L_{4\cdot AP}) \leq 0.1031746$.***

Saad and Wolf (2017) previously established an u.b. of 0.1247 with no non-trivial l.b. known.

**Proposition (Rué and S., 2023)**

*We have $m_{q=3, c=3}(L_{3\cdot AP}) = 1/27$.***

Similar to Cummings et al. (2013) extending a result of Goodman (1959) about triangles.

---

**Upper bounds**

obtained through blow-up constructions of particular finite colorings.

**Lower bounds**

obtained by extending Razborov’s Flag Algebra framework.
The Rado Multiplicity Problem in $\mathbb{F}_q^n$

1. The Rado Multiplicity Problem 3 slides

2. Constructive Upper Bounds through Blow-ups 2 slides

3. Lower Bounds through Flag Algebras 4 slides

4. Concluding Remarks and Open Problems 1 slide
2. Constructive Upper Bounds through Blow-ups

How to blow up colorings

The bound of Saad and Wolf relied on a Fourier-analytic probabilistic construction.

The ‘quantitatively superior’ approach however is to consider blowups:

Sometimes we have a free element * in which we can iterate the blowup-construction:
2. Constructive Upper Bounds through Blow-ups

**How to blow up colorings**

The bound of Saad and Wolf relied on a Fourier-analytic probabilistic construction.

The ‘quantitatively superior’ approach however is to consider *blowups*:

\[
\begin{array}{ccc}
\begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\square & \square & \square \\
\end{array} & \rightarrow & \begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\square & \square & \square \\
\end{array} & \rightarrow & \begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\square & \square & \square \\
\end{array} & \rightarrow & \ldots
\end{array}
\]

Sometimes we have a *free element* \( \ast \) in which we can iterate the blowup-construction:

\[
\begin{array}{ccc}
\begin{array}{ccc}
\ast & \square & \square \\
\square & \square & \square \\
\square & \square & \square \\
\end{array} & \rightarrow & \begin{array}{ccc}
\ast & \square & \square \\
\square & \square & \square \\
\square & \square & \square \\
\end{array} & \rightarrow & \begin{array}{ccc}
\ast & \square & \square \\
\square & \square & \square \\
\square & \square & \square \\
\end{array} & \rightarrow & \ldots
\end{array}
\]
2. Constructive Upper Bounds through Blow-ups

How to blow up colorings

The bound of Saad and Wolf relied on a Fourier-analytic probabilistic construction. The ‘quantitatively superior’ approach however is to consider blowups:

Sometimes we have a free element $*$ in which we can iterate the blowup-construction:
2. Constructive Upper Bounds through Blow-ups

**Proofs of the upper bounds**

**Upper bound of the Theorem**

\[ m_{3,3}(L_{4\text{-AP}}) \leq 1/27 \] follows from the blow-up of this 3-coloring of \( \mathbb{R}^3 \):

```
  |   |   |   |
---|---|---|---|
  |   |   |   |
  |   |   |   |
```

**Upper bound of the Proposition**

\[ m_{5,2}(L_{4\text{-AP}}) \leq 13/126 \] follows from the iterated blow-up of this 2-coloring of \( \mathbb{R}^3 \):

```
*  |   |   |   |   |   |   |   |   |
---|---|---|---|---|---|---|---|---|
  |   |   |   |   |   |   |   |   |
  |   |   |   |   |   |   |   |   |
  |   |   |   |   |   |   |   |   |
  |   |   |   |   |   |   |   |   |
  |   |   |   |   |   |   |   |   |
  |   |   |   |   |   |   |   |   |
  |   |   |   |   |   |   |   |   |
  |   |   |   |   |   |   |   |   |
  |   |   |   |   |   |   |   |   |
```
The Rado Multiplicity Problem in $\mathbb{F}_q^n$

1. The Rado Multiplicity Problem  
   - 3 slides

2. Constructive Upper Bounds through Blow-ups  
   - 2 slides

3. Lower Bounds through Flag Algebras  
   - 4 slides

4. Concluding Remarks and Open Problems  
   - 1 slide
3. Lower Bounds through Flag Algebras

Where are Flag Algebras in Additive Combinatorics?

In the table below we have marked in bold a monochromatic or rainbow arithmetic progression in each 3-coloring of the 9-tuples. This proves that any 3-coloring of any 9-tuple contains a non-degenerate arithmetic progression of length 3 belonging to \( M \) or \( R \).

\[
\begin{array}{cccc}
111** & 1121221* & 12122111* & 1221213** \\
112111** & 11221222* & 121221112* & 12212211** \\
1121121** & 112212231 & 12122113* & 1221222** \\
11211221* & 112212232 & 121221211 & 12212231* \\
1121123* & 112212233 & 121221212 & 12212232* \\
11211233* & 1122123** & 121221213 & 12212233* \\
11211233** & 112213*** & 12122122* & 122123** \\
1121131** & 11222**** & 12122123* & 12213**** \\
1121132** & 11223***** & 1212213** & 1222****** \\
1121133** & 1123****** & 121222*** & 12231***** \\
1121211*** & 11211***** & 1212231** & 122321*** \\
1121221** & 1121211** & 12122321* & 12232211* \\
1121222** & 11212121 & 12122322 & 12232212
\end{array}
\]

3. Lower Bounds through Flag Algebras

Correctly defining our combinatorial structures

**Definition (Partially fixed Morphisms, Monomorphisms, and Isomorphisms)**

An affine linear map $\varphi : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^n$ as a $t$-fixed morphism iff $\varphi(e_j) = e_j$ for all $0 \leq j \leq t$ (where $t \geq -1$ and $e_0 = 0$). It is a mono/isomorphism iff it is in/bijective.

This gives us ...

- ... a notion of isomorphic colorings through isomorphisms,
- ... a notion of substructure or sub-coloring through monomorphisms,
- ... a notion of density,
- ... a notion of a ‘type’ through $t$.

The resulting notion of density crucially satisfies the *averaging equality* 

$$ p(\text{small, large}) = \sum_{\text{medium}} p(\text{small, medium}) \cdot p(\text{medium, large}). \quad (2) $$
3. Lower Bounds through Flag Algebras

**Correctly defining solutions**

**Problem.** How to count solutions through colorings? In $\mathbb{F}_3^n$ for example, the Schur triple $(0, 0, \bar{0}), (1, 2, \bar{0}), (2, 1, \bar{0})$ defines a unique 2-dimensional linear subspace, but the Schur triple $(0, 0, \bar{0}), (1, 1, \bar{0}), (2, 2, \bar{0})$ does not ...

**Definition**

The *dimension* $\dim_t(s)$ of $s \in S_L$ is the smallest dimension of a $t$-fixed subspace containing it and $\dim_t(L)$ is the largest dimension of any solution.

Each fully dimensional solution determines a unique $\dim_t(L)$-dimensional substructure in which it lies. Writing $S^t_L(T) = \{s \in S_L(T) : \dim_t(s) = \dim_t(L)\}$, we have

$$|S^t_L(\mathbb{F}_q^n)| = |S(\mathbb{F}_q^n)| (1 + o(1)).$$

Fully-dimensional solution satisfy an averaging equality like (2).
3. Lower Bounds through Flag Algebras

Correctly defining solutions

**Problem.** How to count solutions through colorings? In $\mathbb{F}_3^n$ for example, the Schur triple $(0, 0, 0), (1, 2, 0), (2, 1, 0)$ defines a unique 2-dimensional linear subspace, but the Schur triple $(0, 0, 0), (1, 1, 0), (2, 2, 0)$ does not ...

**Definition**

The *dimension* $\dim_t(s)$ of $s \in S_L$ is the smallest dimension of a $t$-fixed subspace containing it and $\dim_t(L)$ is the largest dimension of any solution.

Each fully dimensional solution determines a unique $\dim_t(L)$-dimensional substructure in which it lies. Writing $S^t_L(T) = \{s \in S_L(T) : \dim_t(s) = \dim_t(L)\}$, we have

$$|S^t_L(\mathbb{F}_q^n)| = |S(\mathbb{F}_q^n)| (1 + o(1)).$$

Fully-dimensional solution satisfy an averaging equality like (2).
Problem. How to count solutions through colorings? In $\mathbb{F}_3^n$ for example, the Schur triple $(0, 0, 0), (1, 2, 0), (2, 1, 0)$ defines a unique 2-dimensional linear subspace, but the Schur triple $(0, 0, 0), (1, 1, 0), (2, 2, 0)$ does not ...  

**Definition**

The *dimension* $\dim_t(s)$ of $s \in S_L$ is the smallest dimension of a $t$-fixed subspace containing it and $\dim_t(L)$ is the largest dimension of any solution.

Each fully dimensional solution determines a unique $\dim_t(L)$-dimensional substructure in which it lies. Writing $S^t_L(T) = \{s \in S_L(T) : \dim_t(s) = \dim_t(L)\}$, we have

$$|S^t_L(\mathbb{F}_q^n)| = |S(\mathbb{F}_q^n)| (1 + o(1)).$$

**Fully-dimensional solution** satisfy an averaging equality like (2).
3. Lower Bounds through Flag Algebras

**Definition**

The *flag algebra* $\mathcal{A}$ is given by considering linear combinations of colorings, factoring out relations given by the averaging equality (2) and defining an appropriate product.

Razborov established a bijection between sequences $(G_n)$ where all $p(H; G_n)$ converge and $\varphi \in \text{Hom}(\mathcal{A}, \mathbb{R})$ satisfying $\varphi(H) \geq 0$ for all $H \in \mathcal{G}$ through $p(H; G_n) = \varphi(H)$.

The *semantic cone* $S = \{ f \in \mathcal{A} : \phi(f) \geq 0 \text{ for all } \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \}$ captures those algebraic expressions that correspond to density expressions that are ‘true’.

Letting $C_L \in \mathcal{A}$ capturing the behavior of $|S_L^f(\mathbb{F}_q^n)|$, we can establish a lower bound by finding and verifying a sum-of-squares (SOS) expression

$$C_L - \lambda - \sum_{i=1}^{k} f_i^2 \in S. \quad (3)$$
3. Lower Bounds through Flag Algebras

SOS please someone help me

Definition

The *flag algebra* $\mathcal{A}$ is given by considering linear combinations of colorings, factoring out relations given by the averaging equality (2) and defining an appropriate product.

Razborov established a bijection between sequences $(G_n)$ where all $p(H; G_n)$ converge and $\varphi \in \text{Hom}(\mathcal{A}, \mathbb{R})$ satisfying $\varphi(H) \geq 0$ for all $H \in \mathcal{G}$ through $p(H; G_n) = \varphi(H)$.

The *semantic cone* $\mathcal{S} = \{ f \in \mathcal{A} : \phi(f) \geq 0 \text{ for all } \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \}$ captures those algebraic expressions that correspond to density expressions that are ‘true’.

Letting $C_L \in \mathcal{A}$ capturing the behavior of $|S_L^e(\mathbb{F}_q^n)|$, we can establish a lower bound by finding and verifying a sum-of-squares (SOS) expression

$$C_L - \lambda - \sum_{i=1}^{k} f_i^2 \in \mathcal{S}. \quad (3)$$
3. Lower Bounds through Flag Algebras

**Lower bound of the Proposition**

\( m_{5,2}(L_{4-AP}) > 1/10 \) follows by verifying that over all 3324 2-colorings of \( \mathbb{F}_5^2 \) we have

\[
F_1 + F_4 + (F_2 + F_3)/5 - 1/10 \geq \sum_{i=1}^{2} \left( \frac{9}{10} \cdot \left[ (F_{i,1} + (5F_{i,2} - 5F_{i,3} - 10F_{i,4})/27 \right]^2 \right)_{-1} \\
\ldots + \frac{61}{162} \cdot \left[ ((F_{i,3} - F_{i,2})/2 + F_{i,4})^2 \right]_{-1}
\]

and by noting that \( F_{1,1} + F_{2,1} > 0 \). Here the relevant flags \( F_i \) and \( F_{i,j} \) are

<table>
<thead>
<tr>
<th>Flags of type Ø</th>
<th>Flags of type □</th>
<th>Flags of type ■</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>( F_{1,1} )</td>
<td>( F_{2,1} )</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>( F_{1,2} )</td>
<td>( F_{2,2} )</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>( F_{1,3} )</td>
<td>( F_{2,3} )</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>( F_{1,4} )</td>
<td>( F_{2,4} )</td>
</tr>
</tbody>
</table>
3. Lower Bounds through Flag Algebras

**Lower bound of the Theorem**

\[ m_{3,3}(L_{3\text{-AP}}) \geq \frac{1}{27} \] follows by verifying that over all 140 3-colorings of \( \mathbb{F}_3^2 \) we have

\[
F_i - \frac{1}{27} \geq \frac{26}{27} \cdot \left( (F_{i,1} - \frac{99}{182} F_{i,2} + \frac{75}{208} F_{i,3} - \frac{11}{28} F_{i,4} - \frac{3}{26} F_{i,5})^2 \right)_{-1} \\
\quad \quad \quad + \frac{1685}{1911} \cdot \left( (F_{i,2} - \frac{231}{26960} F_{i,3} + \frac{1703}{6740} F_{i,4} - \frac{1869}{3370} F_{i,5})^2 \right)_{-1} \\
\quad \quad \quad + \frac{71779}{431360} \cdot \left( (F_{i,3} - \frac{358196}{502453} F_{i,4} - \frac{412904}{502453} F_{i,5})^2 \right)_{-1} \\
\quad \quad \quad + \frac{5431408}{10551513} \cdot \left( (F_{i,4} - \frac{1}{4} F_{i,5})^2 \right)_{-1}
\]

for any \( i \in \{1, 2, 3\} \). Here the relevant flags \( F_i \) and \( F_{i,j} \) are

<table>
<thead>
<tr>
<th>Flags of type ( \emptyset )</th>
<th>Flags of type ( \blacksquare )</th>
<th>Flags of type ( \square )</th>
<th>Flags of type ( \blacksquare )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>( F_{1,1} )</td>
<td>( F_{2,1} )</td>
<td>( F_{3,1} )</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>( F_{1,2} )</td>
<td>( F_{2,2} )</td>
<td>( F_{3,2} )</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>( F_{1,3} )</td>
<td>( F_{2,3} )</td>
<td>( F_{3,3} )</td>
</tr>
<tr>
<td>( F_{1,4} )</td>
<td>( F_{2,4} )</td>
<td>( F_{3,4} )</td>
<td></td>
</tr>
<tr>
<td>( F_{1,5} )</td>
<td>( F_{2,5} )</td>
<td>( F_{3,5} )</td>
<td></td>
</tr>
</tbody>
</table>
The Rado Multiplicity Problem in $\mathbb{F}_q^n$ 

1. The Rado Multiplicity Problem  
   3 slides 

2. Constructive Upper Bounds through Blow-ups  
   2 slides 

3. Lower Bounds through Flag Algebras  
   4 slides 

4. Concluding Remarks and Open Problems  
   1 slide
Fourth Section: Concluding Remarks and Open Problems

Open Problems and Final Remarks

- Often one can extract stability results from Flag Algebra certificates.
- Steep computational hurdle: underlying structures grow exponentially (instead of quadratically for graphs or cubic for 3-uniform hypergraphs).
- No neat notion of subspaces makes generalizing to other groups difficult.

Code is available at github.com/FordUniver/rs_radomult_23
Thank you for your attention!