Towards Flag Algebras in Additive Combinatorics

FoCM 2023 – Workshop I.3: Graph Theory and Combinatorics

Juanjo Rué Perna and Christoph Spiegel

13th of June 2023
Towards Flag Algebras in Additive Combinatorics

1. The Trouble with Defining Additive Flag Algebras

2. The Rado Multiplicity Problem

3. Proofs of Lower Bounds
1. The Trouble with Defining Additive Flag Algebras

The Motivation

Why are there no Flag Algebras in Additive Combinatorics?

Given $T \subseteq G$ and linear map $L$, we care about

$$S_L(T) \overset{\text{def}}{=} \{ s \in T^m : L(s) = 0, s_i \neq s_j \text{ for } i \neq j \},$$

where $G = [n], \mathbb{Z}_n, \mathbb{Z}_p, \mathbb{F}_q^n, \ldots$ and $L$ represents AP, Schur triples, ...

- Ramsey (1930) $\leftrightarrow$ Schur (1917), van der Waerden (1927), and Rado (1933)
- Mantel (1907) and Turán (1941) $\leftrightarrow$ Roth (1953) and Szemerédi (1975)
- regularity lemma (Szemerédi, 1978) $\leftrightarrow$ arithmetic regularity (Green, 2005)
- random graph $G(n, p)$ $\leftrightarrow$ random sets $[n]_p, (\mathbb{Z}_n)_p, \ldots$
- blowup-type constructions are relevant in both
1. The Trouble with Defining Additive Flag Algebras

The Motivation

Why are there no Flag Algebras in Additive Combinatorics?

Given $T \subset G$ and linear map $L$, we care about

$$S_L(T) \overset{\text{def}}{=} \{s \in T^m : L(s) = \bar{0}, s_i \neq s_j \text{ for } i \neq j\},$$

where $G = [n], \mathbb{Z}_n, \mathbb{Z}_p, \mathbb{F}_q^n, \ldots$ and $L$ represents AP, Schur triples, ...

- Ramsey (1930) $\iff$ Schur (1917), van der Waerden (1927), and Rado (1933)
- Mantel (1907) and Turán (1941) $\iff$ Roth (1953) and Szemerédi (1975)
- regularity lemma (Szemerédi, 1978) $\iff$ arithmetic regularity (Green, 2005)
- random graph $G(n, p) \iff$ random sets $[n]_p, (\mathbb{Z}_n)_p, \ldots$
- blowup-type constructions are relevant in both
1. The Trouble with Defining Additive Flag Algebras

The Motivation

Why are there no Flag Algebras in Additive Combinatorics?

Given $T \subset G$ and linear map $L$, we care about

$$S_L(T) \overset{\text{def}}{=} \{ s \in T^m : L(s) = 0, s_i \neq s_j \text{ for } i \neq j \},$$

where $G = [n], \mathbb{Z}_n, \mathbb{Z}_p, \mathbb{F}_q^n, \ldots$ and $L$ represents AP, Schur triples, ...

- Ramsey (1930) $\iff$ Schur (1917), van der Waerden (1927), and Rado (1933)
- Mantel (1907) and Turán (1941) $\iff$ Roth (1953) and Szemerédi (1975)
- regularity lemma (Szemerédi, 1978) $\iff$ arithmetic regularity (Green, 2005)
- random graph $G(n, p) \iff$ random sets $[n]_p, (\mathbb{Z}_n)_p, \ldots$
- blowup-type constructions are relevant in both
1. The Trouble with Defining Additive Flag Algebras

The Motivation

In the table below we have marked in bold a monochromatic or rainbow arithmetic progression in each 3-coloring of the 9-tuples. This proves that any 3-coloring of any 9-tuple contains a non-degenerate arithmetic progression of length 3 belonging to $M$ or $R$.

| 111***** | 11221221* | 12122111* | 1221213** |
| 112111*** | 11221222* | 12122112* | 1221221** |
| 11211121** | 112212231 | 12122113* | 1221222** |
| 11211221* | 112212232 | 121221211 | 12212231* |
| 11211222* | 112212233 | 121221212 | 12212232* |
| 11211223* | 1122123** | 121221213 | 12212233* |
| 1121123** | 112213*** | 12122122* | 122123*** |
| 1121131** | 11222**** | 12122123* | 12213**** |
| 1121132** | 11223***** | 1212213** | 1222****** |
| 1121133** | 1123****** | 121222*** | 12231***** |
| 112121*** | 12111**** | 1212231** | 122321*** |
| 1121221** | 1211211** | 12122321* | 12232211* |
| 1121233** | 12112312 | 12122322* | 12232312 |

1. The Trouble with Defining Additive Flag Algebras

**A Problem**

What we need is a rule like

\[ p(\text{small, large}) = \sum_{\text{medium}} p(\text{small, medium}) \cdot p(\text{medium, large}) \]  

(2)

for some notion of density

\[ p(\text{struct, } \cdot) = \frac{\# \text{ substructures isomorphic to struct in } \cdot}{\# \text{ substructures of same size as struct in } \cdot} \]  

(3)

Finding a working notion of substructure seems difficult in \([n], \mathbb{Z}_n, \mathbb{Z}_p \ldots\)

Can we formulate Flag Algebras for \(GF(q)^n = \mathbb{F}_q^n\)?
1. The Trouble with Defining Additive Flag Algebras

A Problem

What we need is a rule like

\[
p(\text{small, large}) = \sum_{\text{medium}} p(\text{small, medium}) \cdot p(\text{medium, large})
\]

(2)

for some notion of density

\[
p(\text{struct, } \bullet) = \frac{\# \text{ substructures isomorphic to struct in } \bullet}{\# \text{ substructures of same size as struct in } \bullet}.
\]

(3)

Finding a working notion of substructure seems difficult in \([n], \mathbb{Z}_n, \mathbb{Z}_p \ldots\)

Can we formulate Flag Algebras for \(GF(q)^n = \mathbb{F}_q^n\)?
1. The Trouble with Defining Additive Flag Algebras

**A Problem**

What we need is a rule like

$$p(\text{small, large}) = \sum_{\text{medium}} p(\text{small, medium}) \cdot p(\text{medium, large})$$  \hspace{1cm} (2)

for some notion of density

$$p(\text{struct, } \bullet) = \frac{\# \text{ substructures isomorphic to struct in } \bullet}{\# \text{ substructures of same size as struct in } \bullet}.$$ \hspace{1cm} (3)

Finding a working notion of substructure seems difficult in $[n], \mathbb{Z}_n, \mathbb{Z}_p \ldots$

Can we formulate Flag Algebras for $GF(q)^n = \mathbb{F}_q^n$?
1. The Trouble with Defining Additive Flag Algebras

2. The Rado Multiplicity Problem

3. Proofs of Lower Bounds
2. The Rado Multiplicity Problem

Counting monochromatic solutions

Given a coloring $\gamma : \mathbb{F}_q^n \rightarrow [c]$ and linear map, we are interested in

$$S_L(\gamma) \overset{\text{def}}{=} \bigcup_{i=1}^{c} S_L(\gamma^{-1}(\{i\})).$$

(4)

Rado (1933) tells us that $S_L(\gamma) \neq \emptyset$ for large enough $n$ if $L$ satisfies column condition.

The Rado Multiplicity Problem is concerned with determining

$$m_{q,c}(L) \overset{\text{def}}{=} \lim_{n \rightarrow \infty} \min_{\gamma \in \Gamma(n)} \frac{|S_L(\gamma)|}{|S_L(\mathbb{F}_q^n)|}.$$

Limit exists by monotonicity and $0 < m_{q,c}(L) \leq 1$ if $L$ is partition regular. $L$ is $c$-common if $m_{q,c}(L) = c^{1-m}$ (the value attained in a uniform random coloring).
2. The Rado Multiplicity Problem

**Counting monochromatic solutions**

Given a coloring \( \gamma : \mathbb{F}_q^n \rightarrow [c] \) and linear map, we are interested in

\[
S_L(\gamma) \overset{\text{def}}{=} \bigcup_{i=1}^{c} S_L(\gamma^{-1} \{i\}).
\]

(4)

Rado (1933) tells us that \( S_L(\gamma) \neq \emptyset \) for large enough \( n \) if \( L \) satisfies *column condition*.

The **Rado Multiplicity Problem** is concerned with determining

\[
m_{q,c}(L) \overset{\text{def}}{=} \lim_{n \to \infty} \min_{\gamma \in \Gamma(n)} \min \left| S_L(\gamma) \right| / \left| S_L(\mathbb{F}_q^n) \right|.
\]

Limit exists by monotonicity and \( 0 < m_{q,c}(L) \leq 1 \) if \( L \) is partition regular. \( L \) is **c-common** if \( m_{q,c}(L) = c^{1-m} \) (the value attained in a uniform random coloring).
2. The Rado Multiplicity Problem

Previous results

- Graham et al. (1996) gave lower bound for **Schur triples** in 2-colorings of \([n]\), later independently resolved by Robertson and Zeilberger / Schoen / Datskovsky.

- Cameron et al. (2007) showed that the nr. of solutions for linear equations with **an odd nr. of variables** only depends on cardinalities of the two color classes.

- Parrilo, Robertson and Saracino (2008) established bounds for the minimum number of **monochromatic 3-APs** in 2-colorings of \([n]\) (not 2-common in \(\mathbb{N}\)).

- For \(r = 1\) and \(m\) even, Saad and Wolf (2017) showed that any ‘pair-partitionable’ \(L\) is 2-common in \(\mathbb{F}_q^n\). Fox, Pham, and Zhao (2021) showed that this is necessary.

- Kamčev et al. (2021) characterized some non-common \(L\) in \(\mathbb{F}_q^n\) with \(r > 1\).

- Král et al. (2022) characterized 2-common \(L\) for \(q = 2, r = 2, m\) odd.
2. The Rado Multiplicity Problem

**Previous results**


- Cameron et al. (2007) showed that the nr. of solutions for linear equations with an odd nr. of variables only depends on cardinalities of the two color classes.

- Parrilo, Robertson and Saracino (2008) established bounds for the minimum number of *monochromatic 3-APs* in 2-colorings of $[n]$ (not 2-common in $\mathbb{N}$).

- For $r = 1$ and $m$ even, Saad and Wolf (2017) showed that any ‘pair-partitionable’ $L$ is 2-common in $\mathbb{F}_q^n$. Fox, Pham, and Zhao (2021) showed that this is necessary.

- Kamčev et al. (2021) characterized some non-common $L$ in $\mathbb{F}_q^n$ with $r > 1$.

- Král et al. (2022) characterized 2-common $L$ for $q = 2$, $r = 2$, $m$ odd.
2. The Rado Multiplicity Problem

Our results

**Theorem (Rué and S., 2023)**

We have $\frac{1}{10} < m_{q=5,c=2}(L_{4-AP}) \leq 0.1031746$.

Saad and Wolf (2017) previously established an u.b. of 0.1247 with no non-trivial l.b. known.

**Proposition (Rué and S., 2023)**

We have $m_{q=3,c=3}(L_{3-AP}) = \frac{1}{27}$.

Similar to Cummings et al. (2013) extending a result of Goodman (1959) about triangles.

Proofs are computational:

- **Upper bounds** obtained through (iterated) blow-up constructions found through exhaustive and heuristic searches.

- **Lower bounds** obtained through SOS expressions in Flag Algebras found through an SDP solver.
Towards Flag Algebras in Additive Combinatorics

1. The Trouble with Defining Additive Flag Algebras

2. The Rado Multiplicity Problem

3. Proofs of Lower Bounds
3. Proofs of Lower Bounds

The Right Notion of Substructure

Definition (Partially fixed Morphisms, Monomorphisms, and Isomorphisms)

An affine linear map $\varphi : \mathbb{F}_q^k \to \mathbb{F}_q^n$ as a $t$-fixed morphism iff $\varphi(e_j) = e_j$ for all $0 \leq j \leq t$ (where $t \geq -1$ and $e_0 = 0$). It is a mono/isomorphism iff it is in/bijective.

This gives us ...

- ... a notion of isomorphic colorings through isomorphisms,
- ... a notion of substructure or sub-coloring through monomorphisms,
- ... a notion of density through (3) that satisfies (2),
- ... blow-up bounds through not-necessarily-injective morphisms,
- ... a notion of a ‘type’ through $t$,

Remark

The ‘base’ case is $t = -1$ for invariant structures and $t = 0$ otherwise.
3. Proofs of Lower Bounds

The Right Notion of Substructure

Definition (Partially fixed Morphisms, Monomorphisms, and Isomorphisms)

An affine linear map $\varphi : \mathbb{F}_q^k \to \mathbb{F}_q^n$ as a $t$-fixed morphism iff $\varphi(e_j) = e_j$ for all $0 \leq j \leq t$ (where $t \geq -1$ and $e_0 = 0$). It is a mono/isomorphism iff it is in/bijective.

This gives us ...

- ... a notion of isomorphic colorings through isomorphisms,
- ... a notion of substructure or sub-coloring through monomorphisms,
- ... a notion of density through (3) that satisfies (2),
- ... blow-up bounds through not-necessarily-injective morphisms,
- ... a notion of a ‘type’ through $t$,

Remark

The ‘base’ case is $t = -1$ for invariant structures and $t = 0$ otherwise.
3. Proofs of Lower Bounds

The Right Notion of Substructure

Definition (Partially fixed Morphisms, Monomorphisms, and Isomorphisms)

An affine linear map $\varphi : \mathbb{F}_q^k \to \mathbb{F}_q^n$ as a $t$-fixed morphism iff $\varphi(e_j) = e_j$ for all $0 \leq j \leq t$ (where $t \geq -1$ and $e_0 = 0$). It is a mono/isomorphism iff it is in/bijective.

This gives us ...

- ... a notion of isomorphic colorings through isomorphisms,
- ... a notion of substructure or sub-coloring through monomorphisms,
- ... a notion of density through (3) that satisfies (2),
- ... blow-up bounds through not-necessarily-injective morphisms,
- ... a notion of a ‘type’ through $t$,

Remark

*The ‘base’ case is $t = -1$ for invariant structures and $t = 0$ otherwise.*
3. Proofs of Lower Bounds

**Counting solutions through colorings**

**Problem.** How to count solutions through colorings? In $\mathbb{F}_3^n$ for example, the Schur triple $(0, 0, \bar{0}), (1, 2, \bar{0}), (2, 1, \bar{0})$ defines a unique 2-dimensional linear subspace, but the Schur triple $(0, 0, \bar{0}), (1, 1, 0), (2, 2, \bar{0})$ does not ...

**Definition**

The *dimension* $\dim_t(s)$ of $s \in S_L$ is the smallest dimension of a $t$-fixed subspace containing it and $\dim_t(L)$ is the largest dimension of any solution.

Each fully dimensional solution determines a unique $\dim_t(L)$-dimensional substructure in which it lies. Writing $S^t_L(T) = \{s \in S_L(T) : \dim_t(s) = \dim_t(L)\}$, we have

$$|S^t_L(\mathbb{F}_q^n)| = |S(\mathbb{F}_q^n)| (1 + o(1)).$$

So fully-dimensional solutions is what we are *actually* counting!
3. Proofs of Lower Bounds

Counting solutions through colorings

Problem. How to count solutions through colorings? In $\mathbb{F}_3^n$ for example, the Schur triple $(0, 0, 0), (1, 2, 0), (2, 1, 0)$ defines a unique 2-dimensional linear subspace, but the Schur triple $(0, 0, 0), (1, 1, 0), (2, 2, 0)$ does not ...

Definition

The dimension $\dim_t(s)$ of $s \in S_L$ is the smallest dimension of a $t$-fixed subspace containing it and $\dim_t(L)$ is the largest dimension of any solution.

Each fully dimensional solution determines a unique $\dim_t(L)$-dimensional substructure in which it lies. Writing $S^t_L(T) = \{s \in S_L(T) : \dim_t(s) = \dim_t(L)\}$, we have

$$|S^t_L(\mathbb{F}_q^n)| = |S(\mathbb{F}_q^n)| (1 + o(1)).$$

So fully-dimensional solutions is what we are actually counting!
3. Proofs of Lower Bounds

**Counting solutions through colorings**

**Problem.** How to count solutions through colorings? In \( \mathbb{F}_3^n \) for example, the Schur triple \((0, 0, 0), (1, 2, 0), (2, 1, 0)\) defines a unique 2-dimensional linear subspace, but the Schur triple \((0, 0, 0), (1, 1, 0), (2, 2, 0)\) does not ...

**Definition**

The *dimension* \( \dim_t(s) \) of \( s \in S_L \) is the smallest dimension of a \( t \)-fixed subspace containing it and \( \dim_t(L) \) is the largest dimension of any solution.

Each fully dimensional solution determines a unique \( \dim_t(L) \)-dimensional substructure in which it lies. Writing \( S^t_L(T) = \{ s \in S_L(T) : \dim_t(s) = \dim_t(L) \} \), we have

\[
|S^t_L(\mathbb{F}_q^n)| = |S(\mathbb{F}_q^n)| (1 + o(1)).
\]

So fully-dimensional solutions is what we are *actually* counting!
3. Proofs of Lower Bounds

**SOS please someone help me**

**Definition**

The (unfixed or 0-fixed) flag algebra $\mathcal{A}$ is given by considering linear combinations of (unfixed or 0-fixed) colorings, factoring out (2) and defining an appropriate product.

The **semantic cone**

$$S = \{ f \in \mathcal{A} : \phi(f) \geq 0 \text{ for all } \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \}$$  \hspace{1cm} (5)

captures those algebraic expressions corresponding to density expressions that are ‘true’. We can establish a lower bound through

$$C_L - \lambda - \sum_{i=1}^{k} (f_i)^2 \in S,$$  \hspace{1cm} (6)

where $C_L \in \mathcal{A}$ counts fully-dimensional solutions. Such sum-of-squares (SOS) expressions are solvable through Semidefinite Programming (SDP).
3. Proofs of Lower Bounds

**SOS please someone help me**

**Definition**

The (unfixed or 0-fixed) *flag algebra* $\mathcal{A}$ is given by considering linear combinations of (unfixed or 0-fixed) colorings, factoring out (2) and defining an appropriate product.

The *semantic cone*

$$S = \{ f \in \mathcal{A} : \phi(f) \geq 0 \text{ for all } \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \} \quad (5)$$

captures those algebraic expressions corresponding to density expressions that are ‘true’. We can establish a lower bound through

$$C_L - \lambda - \sum_{i=1}^{k} (f_i)^2 \in S, \quad (6)$$

where $C_L \in \mathcal{A}$ counts fully-dimensional solutions. Such sum-of-squares (SOS) expressions are solvable through Semidefinite Programming (SDP).
Definition

The (unfixed or 0-fixed) flag algebra $\mathcal{A}$ is given by considering linear combinations of (unfixed or 0-fixed) colorings, factoring out (2) and defining an appropriate product.

The semantic cone

$$S = \{ f \in \mathcal{A} : \phi(f) \geq 0 \text{ for all } \phi \in \text{Hom}^+ (\mathcal{A}, \mathbb{R}) \} \quad (5)$$

captures those algebraic expressions corresponding to density expressions that are ‘true’. We can establish a lower bound through

$$C_L - \lambda - \sum_{i=1}^{k} (f_i)^2 \in S, \quad (6)$$

where $C_L \in \mathcal{A}$ counts fully-dimensional solutions. Such sum-of-squares (SOS) expressions are solvable through Semidefinite Programming (SDP).
3. Proofs of Lower Bounds

**Lower bound of the Proposition**

\[ m_{5,2}(L_{4-\text{AP}}) > 1/10 \] follows by verifying that

\[
F_1 + F_4 + (F_2 + F_3)/5 - 1/10 \geq \sum_{i=1}^{2} \left( 9/10 \cdot \left( (F_{i,1} + (5F_{i,2} - 5F_{i,3} - 10F_{i,4})/27 \right)^2 \right)_{-1} \\
\ldots + 61/162 \cdot \left( (F_{i,3} - F_{i,2})/2 + F_{i,4} \right)^2_{-1},
\]

over all 3324 2-colorings of \( \mathbb{F}_5^2 \) (and by noting that \( F_{1,1} + F_{2,1} > 0 \)), where

<table>
<thead>
<tr>
<th>Flags of type ( \emptyset )</th>
<th>Flags of type ( \square )</th>
<th>Flags of type ( \blacksquare )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>( F_{1,1} )</td>
<td>( F_{2,1} )</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>( F_{1,2} )</td>
<td>( F_{2,2} )</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>( F_{1,3} )</td>
<td>( F_{2,3} )</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>( F_{1,4} )</td>
<td>( F_{2,4} )</td>
</tr>
</tbody>
</table>
### 3. Proofs of Lower Bounds

#### Lower bound of the Theorem

\( m_{3,3}(L_{3-AP}) \geq 1/27 \) follows by verifying that

\[
F_i - 1/27 \geq 26/27 \cdot \left[ (F_{i,1} - 99/182 F_{i,2} + 75/208 F_{i,3} - 11/28 F_{i,4} - 3/26 F_{i,5})^2 \right]_{-1} \\
\quad \quad \quad + 1685/1911 \cdot \left[ (F_{i,2} - 231/26960 F_{i,3} + 1703/6740 F_{i,4} - 1869/3370 F_{i,5})^2 \right]_{-1} \\
\quad \quad \quad + 71779/431360 \cdot \left[ (F_{i,3} - 358196/502453 F_{i,4} - 412904/502453 F_{i,5})^2 \right]_{-1} \\
\quad \quad \quad + 5431408/10551513 \cdot \left[ (F_{i,4} - 1/4 F_{i,5})^2 \right]_{-1}
\]

for any \( i \in \{1, 2, 3\} \) over all 140 3-colorings of \( \mathbb{F}_3^2 \), where

<table>
<thead>
<tr>
<th>Flags of type ( \emptyset )</th>
<th>Flags of type ( \square )</th>
<th>Flags of type ( \blacksquare )</th>
<th>Flags of type ( \text{■} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>( F_{1,1} )</td>
<td>( F_{2,1} )</td>
<td>( F_{3,1} )</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>( F_{1,2} )</td>
<td>( F_{2,2} )</td>
<td>( F_{3,2} )</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>( F_{1,3} )</td>
<td>( F_{2,3} )</td>
<td>( F_{3,3} )</td>
</tr>
<tr>
<td>( F_{1,4} )</td>
<td>( F_{2,4} )</td>
<td>( F_{3,4} )</td>
<td></td>
</tr>
<tr>
<td>( F_{1,5} )</td>
<td>( F_{2,5} )</td>
<td>( F_{3,5} )</td>
<td></td>
</tr>
</tbody>
</table>
3. Proofs of Lower Bounds

Final Remarks

- Often one can extract stability results from Flag Algebra certificates.
- Steep computational hurdle: underlying structures grow exponentially.
- No neat notion of subspaces makes generalizing to $[n] / \mathbb{Z}_n / \mathbb{Z}_p$ difficult.

Code is available at github.com/FordUniver/rs_radomult_23
Thank you for your attention!
4. Appendix

**How many colorings are there?**

<table>
<thead>
<tr>
<th>$q/n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>$q/n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>10</td>
<td>32</td>
<td>382</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>20</td>
<td>92</td>
<td>2744</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>14</td>
<td>1028</td>
<td></td>
<td></td>
<td>3</td>
<td>6</td>
<td>36</td>
<td>15636</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>1648</td>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>14</td>
<td>7724</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>3324</td>
<td></td>
<td></td>
<td></td>
<td>5</td>
<td>12</td>
<td>72192</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Number of 2-colorings of $\mathbb{F}_q^n$ up to unfixed (left) and 0-fixed (right) isomorphism.

<table>
<thead>
<tr>
<th>$q/n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>2</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>15</td>
<td>60</td>
<td>996</td>
<td>2</td>
<td>9</td>
<td>30</td>
<td>180</td>
<td>6546</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>140</td>
<td>25665178</td>
<td></td>
<td>3</td>
<td>18</td>
<td>648</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>1630868</td>
<td></td>
<td></td>
<td>4</td>
<td>69</td>
<td>8451708</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>24</td>
<td>70793574</td>
<td></td>
<td></td>
<td>5</td>
<td>72</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Number of 3-colorings of $\mathbb{F}_q^n$ up to unfixed (left) and 0-fixed (right) isomorphism.
4. Appendix

How to blow up colorings

We can *blow up* an colorings into a sequence of colorings with $n$ tending to infinity.

\[
\begin{array}{c}
\begin{array}{c}
\square \square \black
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\square \square \black
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\square \square \black
\end{array}
\end{array}
\rightarrow
\ldots
\end{array}
\]

Computing the density of solutions in the limit of this sequence is easy: simply check *not-necessarily-injective* subcolorings in the base construction. *This gives us an immediate upper bound from any coloring we can come up with ...*

In some cases we have a *free element* in which we can iterate the blowup-construction.

\[
\begin{array}{c}
\begin{array}{c}
\black \square \black
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\black \square \black
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\black \square \black
\end{array}
\end{array}
\rightarrow
\ldots
\end{array}
\]
4. Appendix

Proofs of the upper bounds

Upper bound of the Proposition

\[ m_{5,2}(L_{4-\text{AP}}) \leq \frac{13}{126} \] follows from the iterated blow-up of this 2-coloring of \( \mathbb{F}_5^3 \):

![2-coloring of F5^3](image)

Upper bound of the Theorem

\[ m_{3,3}(L_{4-\text{AP}}) \leq \frac{1}{27} \] follows from the blow-up of this 3-coloring of \( \mathbb{F}_3^3 \):

![3-coloring of F3^3](image)
4. Appendix

Counting Monomorphisms

We write \([n]_q = \sum_{i=0}^{n-1} q^i\) for the \(q\)-number of \(n\), \([n]_q! = [n]_q \cdots [2]_q [1]_q\) for the \(q\)-factorial of \(n\), and let the Gaussian multinomial coefficient be

\[
\binom{n}{k_1, \ldots, k_m}_q = \frac{[n]_q!}{[k_1]_q! \cdots [k_m]_q! [n - k']_q!}.
\]

Lemma (Double Counting)

We have

\[
|\text{Mon}_t(k_1, \ldots, k_m; n')| \cdot |\text{Mon}_t(n'; n)| = |\text{Mon}_t(k_1, \ldots, k_m; n)| \binom{n-k'}{n' - k'}_q
\]

for any \(t \geq -1\), \(k_1, \ldots, k_m \geq t^+\), and \(n \geq n' \geq k' = k_1 + \ldots + k_m - (m - 1) t^+\).
4. Appendix

Counting Monomorphisms

**Lemma (Unfixed Monomorphisms)**

For any integers $0 \leq k_1, \ldots, k_m$ and $n \geq k' = k_1 + \ldots + k_m$, we have

$$|\text{Mon}_- (k_1, \ldots, k_m; n)| = q^{n-k'} \binom{n}{k_1, \ldots, k_m}_q.$$

**Lemma (Fixed Monomorphisms)**

For integers $0 \leq t \leq k_1, \ldots, k_m$ and $n \geq k' = k_1 + \ldots + k_m - (m-1)t$, we have

$$|\text{Mon}_t (k_1, \ldots, k_m; n)| = \binom{n-t}{k_1-t, \ldots, k_m-t}_q.$$