Definition (Maker-Breaker Games)

1. Let $\mathcal{F}$ be hypergraph. In the Maker-Breaker game played on $\mathcal{F}$ there are two players, Maker and Breaker, alternately picking elements of $V(\mathcal{F})$. 
**Definition (Maker-Breaker Games)**

1. Let $F$ be hypergraph. In the *Maker-Breaker game played on* $F$ there are two players, *Maker* and *Breaker*, alternately picking elements of $V(F)$. Maker wins if he completes a winning set $F \in F$. Breaker wins if he can keep Maker from achieving this goal.

**Theorem (Erdős-Selfridge '73, Beck '82)**

If \[ \sum_{F \in F} (1 + q) - |F| < \frac{1}{1+q} \] then the game is a Breaker's win and the winning strategy is given by an efficient deterministic algorithm. There is also a much weaker, rarely used Maker's criterion due to Beck.
Definition (Maker-Breaker Games)

1. Let \( \mathcal{F} \) be hypergraph. In the Maker-Breaker game played on \( \mathcal{F} \) there are two players, *Maker* and *Breaker*, alternately picking elements of \( V(\mathcal{F}) \). Maker wins if he completes a winning set \( F \in \mathcal{F} \). Breaker wins if he can keep Maker from achieving this goal.
Definition (Maker-Breaker Games)

1. Let $\mathcal{F}$ be hypergraph. In the Maker-Breaker game played on $\mathcal{F}$ there are two players, Maker and Breaker, alternately picking elements of $V(\mathcal{F})$. Maker wins if he completes a winning set $F \in \mathcal{F}$. Breaker wins if he can keep Maker from achieving this goal.

2. In the biased Maker-Breaker game, Breaker is allowed to pick $q \geq 1$ elements each turn.
Definition (Maker-Breaker Games)

1. Let $F$ be hypergraph. In the Maker-Breaker game played on $F$ there are two players, Maker and Breaker, alternately picking elements of $V(F)$. Maker wins if he completes a winning set $F \in F$. Breaker wins if he can keep Maker from achieving this goal.

2. In the biased Maker-Breaker game, Breaker is allowed to pick $q \geq 1$ elements each turn. The bias threshold is the value $q_0$ such that Breaker has a winning strategy for $q \geq q_0$ and does not for $q < q_0$. 
Definition (Maker-Breaker Games)

1. Let $\mathcal{F}$ be hypergraph. In the Maker-Breaker game played on $\mathcal{F}$ there are two players, Maker and Breaker, alternately picking elements of $V(\mathcal{F})$. Maker wins if he completes a winning set $F \in \mathcal{F}$. Breaker wins if he can keep Maker from achieving this goal.

2. In the biased Maker-Breaker game, Breaker is allowed to pick $q \geq 1$ elements each turn. The bias threshold is the value $q_0$ such that Breaker has a winning strategy for $q \geq q_0$ and does not for $q < q_0$.

Theorem (Erdős-Selfridge ’73, Beck ’82)

If \[ \sum_{F \in \mathcal{F}} (1 + q)^{-|F|} < 1/(1 + q) \] then the game is a Breaker’s win and the winning strategy is given by an efficient deterministic algorithm.
Definition (Maker-Breaker Games)

1. Let $\mathcal{F}$ be hypergraph. In the Maker-Breaker game played on $\mathcal{F}$ there are two players, Maker and Breaker, alternately picking elements of $V(\mathcal{F})$. Maker wins if he completes a winning set $F \in \mathcal{F}$. Breaker wins if he can keep Maker from achieving this goal.

2. In the biased Maker-Breaker game, Breaker is allowed to pick $q \geq 1$ elements each turn. The bias threshold is the value $q_0$ such that Breaker has a winning strategy for $q \geq q_0$ and does not for $q < q_0$.

Theorem (Erdős-Selfridge ’73, Beck ’82)

If $\sum_{F \in \mathcal{F}} (1 + q)^{-|F|} < 1/(1 + q)$ then the game is a Breaker’s win and the winning strategy is given by an efficient deterministic algorithm.

There is also a much weaker, rarely used Maker’s criterion due to Beck.
Example (Connectivity Game)

The board of the connectivity game is $E(K_n)$ and the winning sets consist of all connected spanning subgraphs of $K_n$. 
Example (Connectivity Game)

The board of the *connectivity game* is $E(K_n)$ and the winning sets consist of all connected spanning subgraphs of $K_n$. There is a simple explicit winning strategy for Maker for all $n$. 
Example (Connectivity Game)

The board of the *connectivity game* is $E(K_n)$ and the winning sets consist of all connected spanning subgraphs of $K_n$. There is a simple explicit winning strategy for Maker for all $n$. The bias threshold satisfies $q_0 = \Theta(n / \ln n)$. 
Example (Connectivity Game)

The board of the *connectivity game* is $E(K_n)$ and the winning sets consist of all connected spanning subgraphs of $K_n$. There is a simple explicit winning strategy for Maker for all $n$. The bias threshold satisfies $q_0 = \Theta(n/\ln n)$.

Example (Triangle Game)

The board of the *triangle game* is $E(K_n)$ and the winning sets are all triangles.
Example (Connectivity Game)
The board of the *connectivity game* is $E(K_n)$ and the winning sets consist of all connected spanning subgraphs of $K_n$. There is a simple explicit winning strategy for Maker for all $n$. The bias threshold satisfies $q_0 = \Theta(n/\ln n)$.

Example (Triangle Game)
The board of the *triangle game* is $E(K_n)$ and the winning sets are all triangles. Simple explicit strategies show that the bias threshold satisfies $q_0 = \Theta(n^{1/2})$. 


Example (Connectivity Game)

The board of the *connectivity game* is $E(K_n)$ and the winning sets consist of all connected spanning subgraphs of $K_n$. There is a simple explicit winning strategy for Maker for all $n$. The bias threshold satisfies $q_0 = \Theta(n/\ln n)$.

Example (Triangle Game)

The board of the *triangle game* is $E(K_n)$ and the winning sets are all triangles. Simple explicit strategies show that the bias threshold satisfies $q_0 = \Theta(n^{1/2})$.

Example (van der Waerden Game – Beck ’81)

*Van der Waerden games* are the positional games played on the board $[n] = \{1, \ldots, n\}$ with all $k$-AP as winning sets.
Van der Waerden Games

Definition (Beck ‘81)
For a given $k \geq 3$ let $W^*(k)$ denote the smallest integer $n$ for which Maker has a winning strategy in the respective van der Waerden game.
Van der Waerden Games

Definition (Beck ‘81)
For a given $k \geq 3$ let $W^*(k)$ denote the smallest integer $n$ for which Maker has a winning strategy in the respective van der Waerden game.

Remark
Let $W(k)$ denote the van der Waerden Number. By van der Waerden’s Theorem Breaker must occupy a $k$-AP for himself if he wants to win. A standard strategy stealing argument therefore gives us $W^*(k) \leq W(k)$. 
Definition (Beck ‘81)
For a given $k \geq 3$ let $W^*(k)$ denote the smallest integer $n$ for which Maker has a winning strategy in the respective van der Waerden game.

Remark
Let $W(k)$ denote the van der Waerden Number. By van der Waerden’s Theorem Breaker must occupy a $k$-AP for himself if he wants to win. A standard strategy stealing argument therefore gives us $W^*(k) \leq W(k)$.

Theorem (Beck ‘81)
We have $W^*(k) = 2^{k(1+o(1))}$. 
Van der Waerden Games

Definition (Beck ‘81)
For a given $k \geq 3$ let $W^*(k)$ denote the smallest integer $n$ for which Maker has a winning strategy in the respective van der Waerden game.

Remark
Let $W(k)$ denote the van der Waerden Number. By van der Waerden’s Theorem Breaker must occupy a $k$-AP for himself if he wants to win. A standard strategy stealing argument therefore gives us $W^*(k) \leq W(k)$.

Theorem (Beck ‘81)
We have $W^*(k) = 2^{k(1+o(1))}$.

What about the biased version?
Proposition

The threshold bias of the 3-AP game played on \([n]\) satisfies

\[
\sqrt{\frac{n}{12}} - \frac{1}{6} \leq q_0(n) \leq \sqrt{3n}.
\]
Proposition

The threshold bias of the 3-AP game played on \([n]\) satisfies

\[
\sqrt{\frac{n}{12} - \frac{1}{6}} \leq q_0(n) \leq \sqrt{3n}.
\]

Proof.

Breaker.
Van der Waerden Games

Proposition

*The threshold bias of the 3-AP game played on \([n]\) satisfies*

\[
\sqrt{\frac{n}{12} - \frac{1}{6}} \leq q_0(n) \leq \sqrt{3n}.
\]

Proof.

**Breaker.** At round \(i\) Breaker covers all \(3(i - 1)\) possibilities that Maker could combine his previous move with any of his other moves in order to form a 3-AP.
Van der Waerden Games

Proposition

*The threshold bias of the 3-AP game played on* \([n]\) *satisfies*

\[
\sqrt{\frac{n}{12} - \frac{1}{6}} \leq q_0(n) \leq \sqrt{3n}.
\]

Proof.

**Breaker.** At round \(i\) Breaker covers all \(3(i - 1)\) possibilities that Maker could combine his previous move with any of his other moves in order to form a 3-AP. Since \(i \leq n/(q + 1)\) Breaker can do so if \(q(q + 1) \geq 3n\), which is the case if \(q \geq \sqrt{3n}\).
Van der Waerden Games

Proposition

The threshold bias of the 3-AP game played on \([n]\) satisfies

\[
\sqrt{\frac{n}{12} - \frac{1}{6}} \leq q_0(n) \leq \sqrt{3n}.
\]

Proof.

**Breaker.** At round \(i\) Breaker covers all \(3(i - 1)\) possibilities that Maker could combine his previous move with any of his other moves in order to form a 3-AP. Since \(i \leq n/(q + 1)\) Breaker can do so if \(q(q + 1) \geq 3n\), which is the case if \(q \geq \sqrt{3n}\).

**Maker.**
Van der Waerden Games

Proposition

The threshold bias of the 3-AP game played on \([n]\) satisfies

\[
\sqrt{\frac{n}{12} - \frac{1}{6}} \leq q_0(n) \leq \sqrt{3n}.
\]

Proof.

**Breaker.** At round \(i\) Breaker covers all \(3(i - 1)\) possibilities that Maker could combine his previous move with any of his other moves in order to form a 3-AP. Since \(i \leq n/(q + 1)\) Breaker can do so if \(q(q + 1) \geq 3n\), which is the case if \(q \geq \sqrt{3n}\).

**Maker.** Use Beck’s Maker criterion.
Van der Waerden Games

Proposition

The threshold bias of the $3$-AP game played on $[n]$ satisfies

$\sqrt{\frac{n}{12}} - \frac{1}{6} \leq q_0(n) \leq \sqrt{3n}$.

Proof.

**Breaker.** At round $i$ Breaker covers all $3(i - 1)$ possibilities that Maker could combine his previous move with any of his other moves in order to form a $3$-AP. Since $i \leq n/(q + 1)$ Breaker can do so if $q(q + 1) \geq 3n$, which is the case if $q \geq \sqrt{3n}$.

**Maker.** Use Beck’s Maker criterion.

What about more general additive structures?
Van der Waerden Games

Definition (Abundant Matrices)

Given some matrix $A \in \mathbb{Z}^{r \times m}$ we call it
Definition (Abundant Matrices)
Given some matrix $A \in \mathbb{Z}^{r \times m}$ we call it

(i) *positive* if there are solutions whose entries are positive,
Definition (Abundant Matrices)
Given some matrix $A \in \mathbb{Z}^{r \times m}$ we call it

(i) *positive* if there are solutions whose entries are positive,

(ii) *abundant* if every submatrix obtained from $A$ by deleting two columns has the same rank as $A$. 
Van der Waerden Games

Definition (Abundant Matrices)
Given some matrix $A \in \mathbb{Z}^{r \times m}$ we call it

(i) *positive* if there are solutions whose entries are positive,

(ii) *abundant* if every submatrix obtained from $A$ by deleting two columns has the same rank as $A$.

Definition (Maximum 1-density)
For $\emptyset \subseteq Q \subseteq [m]$, let $A^Q$ denote the matrix keeping only columns indexed by $Q$ and let $r_Q = \text{rk}(A) - \text{rk}(A^{\overline{Q}})$. 

Van der Waerden Games

Definition (Abundant Matrices)
Given some matrix $A \in \mathbb{Z}^{r \times m}$ we call it

(i) *positive* if there are solutions whose entries are positive,

(ii) *abundant* if every submatrix obtained from $A$ by deleting two columns has the same rank as $A$.

Definition (Maximum 1-density)
For $\emptyset \subseteq Q \subseteq [m]$, let $A^Q$ denote the matrix keeping only columns indexed by $Q$ and let $r_Q = \text{rk}(A) - \text{rk}(A^Q)$. The maximum 1-density of an abundant matrix $A \in \mathbb{Z}^{r \times m}$ is defined as

$$m_1(A) = \max_{Q \subseteq [m]} \frac{|Q| - 1}{|Q| - r_Q - 1}. \quad (1)$$
Example (Schur triple)
The matrix associated with Schur triple is given by

$$A_{\text{Schur}} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \in \mathbb{Z}^{1 \times 3}. \tag{2}$$
Example (Schur triple)

The matrix associated with Schur triple is given by

\[ A_{\text{Schur}} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \in \mathbb{Z}^{1 \times 3}. \]  

(2)

\[ A_{\text{Schur}} \] is abundant and we have \( m_1(A_{\text{Schur}}) = 2. \)
Example (Schur triple)
The matrix associated with Schur triple is given by

\[ A_{\text{Schur}} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \in \mathbb{Z}^{1 \times 3}. \]  

(2)

\( A_{\text{Schur}} \) is abundant and we have \( m_1(A_{\text{Schur}}) = 2. \)

Example (Arithmetic Progressions)
The matrix associated with a \( k \)-term arithmetic progression is given by

\[ A_{k-\text{AP}} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ \vdots \\ 1 & -2 & 1 \end{pmatrix} \in \mathbb{Z}^{(k-2) \times k}. \]  

(3)

Remark (Density and Partition Regularity)
\( (1 \ 1 \ -2) \) is density regular and \( (1 \ 1 \ -1) \) is partition regular.
However, \( (1 \ 1 \ -3) \) is abundant but neither density nor partition regular.
Example (Schur triple)
The matrix associated with Schur triple is given by

\[ A_{\text{Schur}} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \in \mathbb{Z}^{1 \times 3}. \]  

(2)

\( A_{\text{Schur}} \) is abundant and we have \( m_1(A_{\text{Schur}}) = 2 \).

Example (Arithmetic Progressions)
The matrix associated with a \( k \)-term arithmetic progression is given by

\[
A_{k-\text{AP}} = \begin{pmatrix}
1 & -2 & 1 \\
1 & -2 & 1 \\
& & \\
& & \\
& & \\
1 & -2 & 1
\end{pmatrix} \in \mathbb{Z}^{(k-2) \times k}.
\]

(3)

\( A_{k-\text{AP}} \) is abundant and we have \( m_1(A_{k-\text{AP}}) = k - 1 \).
Example (Schur triple)
The matrix associated with Schur triple is given by

\[ A_{\text{Schur}} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \in \mathbb{Z}^{1 \times 3}. \]  

(2)

\( A_{\text{Schur}} \) is abundant and we have \( m_1(A_{\text{Schur}}) = 2. \)

Example (Arithmetic Progressions)
The matrix associated with a \( k \)-term arithmetic progression is given by

\[ A_{k-\text{AP}} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ \vdots \\ 1 & -2 & 1 \end{pmatrix} \in \mathbb{Z}^{(k-2) \times k}. \]  

(3)

\( A_{k-\text{AP}} \) is abundant and we have \( m_1(A_{k-\text{AP}}) = k - 1. \)

Remark (Density and Partition Regularity)
(1 1 −2) is density regular and (1 1 −1) is partition regular.
Example (Schur triple)
The matrix associated with Schur triple is given by

\[ A_{\text{Schur}} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \in \mathbb{Z}^{1 \times 3}. \]  \hspace{1cm} (2)

\( A_{\text{Schur}} \) is abundant and we have \( m_1(A_{\text{Schur}}) = 2. \)

Example (Arithmetic Progressions)
The matrix associated with a \( k \)-term arithmetic progression is given by

\[ A_{k-\text{AP}} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ \vdots \\ 1 & -2 & 1 \end{pmatrix} \in \mathbb{Z}^{(k-2) \times k}. \]  \hspace{1cm} (3)

\( A_{k-\text{AP}} \) is abundant and we have \( m_1(A_{k-\text{AP}}) = k - 1. \)

Remark (Density and Partition Regularity)
\( (1 \ 1 \ -2) \) is density regular and \( (1 \ 1 \ -1) \) is partition regular. However, \( (1 \ 1 \ -3) \) is abundant but neither density nor partition regular.
van der Waerden Games

Definition
Given any matrix $A \in \mathbb{Z}^{r \times m}$ let the corresponding Rado Game be the Maker-Breaker positional game with $[n]$ as the board where the winning sets are $\{x \in [n]^m : A \cdot x^T = 0^T, x_i \neq x_j\}$.
van der Waerden Games

Definition
Given any matrix $A \in \mathbb{Z}^{r \times m}$ let the corresponding Rado Game be the Maker-Breaker positional game with $[n]$ as the board where the winning sets are $\{ x \in [n]^m : A \cdot x^T = 0^T, x_i \neq x_j \}$.

Theorem (Kusch, Rué, S. and Szabó ’17)
For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ the bias threshold of the above game satisfies $q_0(n) = \Theta(n^{1/m_1(A)})$. 
van der Waerden Games

Definition
Given any matrix $A \in \mathbb{Z}^{r \times m}$ let the corresponding Rado Game be the Maker-Breaker positional game with $[n]$ as the board where the winning sets are $\{x \in [n]^m : A \cdot x^T = 0^T, x_i \neq x_j\}$.

Theorem (Kusch, Rué, S. and Szabó ’17)
For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ the bias threshold of the above game satisfies $q_0(n) = \Theta(n^{1/m_1(A)})$. The bias threshold of positive but non-abundant matrices satisfies $q_0(n) \leq 2$. 
van der Waerden Games

Definition
Given any matrix $A \in \mathbb{Z}^{r \times m}$ let the corresponding Rado Game be the Maker-Breaker positional game with $[n]$ as the board where the winning sets are $\{x \in [n]^m : A \cdot x^T = 0^T, \ x_i \neq x_j\}$.

Theorem (Kusch, Rué, S. and Szabó ’17)
For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ the bias threshold of the above game satisfies $q_0(n) = \Theta(n^{1/m_1(A)})$. The bias threshold of positive but non-abundant matrices satisfies $q_0(n) \leq 2$.

Corollary
For van der Waerden games the threshold is $\Theta(n^{1/(k-1)})$ for $k \geq 3$. 

van der Waerden Games

Definition
Given any matrix $A \in \mathbb{Z}^{r \times m}$ let the corresponding Rado Game be the Maker-Breaker positional game with $[n]$ as the board where the winning sets are $\{x \in [n]^m : A \cdot x^T = 0^T, x_i \neq x_j\}$.

Theorem (Kusch, Rué, S. and Szabó ’17)
For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ the bias threshold of the above game satisfies $q_0(n) = \Theta(n^{1/m(A)})$. The bias threshold of positive but non-abundant matrices satisfies $q_0(n) \leq 2$.

Corollary
For van der Waerden games the threshold is $\Theta(n^{1/(k-1)})$ for $k \geq 3$.

There are also a results allowing some repeated entries and results dealing with the inhomogeneous case.
Proof Outline

Bednarska and Łuczak ’00 studied the bias threshold of the Maker-Breaker game consisting of all copies of a given small graph G in $K_n$. 
Proof Outline

Bednarska and Łuczak ’00 studied the bias threshold of the Maker-Breaker game consisting of all copies of a given small graph $G$ in $K_n$.

1. Maker’s strategy is obtained by playing randomly and applying a Lemma on random graphs due to Janson, Łuczak and Ruciński.
Bednarska and Łuczak ’00 studied the bias threshold of the Maker-Breaker game consisting of all copies of a given small graph $G$ in $K_n$.

1. Maker’s strategy is obtained by playing randomly and applying a Lemma on random graphs due to Janson, Łuczak and Ruciński.

2. Breaker’s strategy is obtained by splitting up the bias and simultaneously following multiple strategies given by the Erdős-Selfridge criterion to avoid ‘clustering’.
Proof Outline

Bednarska and Łuczak ’00 studied the bias threshold of the Maker-Breaker game consisting of all copies of a given *small* graph $G$ in $K_n$.

1. Maker’s strategy is obtained by playing randomly and applying a Lemma on random graphs due to Janson, Łuczak and Ruciński.

2. Breaker’s strategy is obtained by splitting up the bias and simultaneously following multiple strategies given by the Erdős-Selfridge criterion to avoid ‘clustering’.

In our paper we extend the ideas behind their proof to obtain general Maker and Breaker Win Criteria and apply them to the Rado games.
Bednarska and Łuczak ’00 studied the bias threshold of the Maker-Breaker game consisting of all copies of a given small graph $G$ in $K_n$.

1. Maker’s strategy is obtained by playing randomly and applying a Lemma on random graphs due to Janson, Łuczak and Ruciński.

2. Breaker’s strategy is obtained by splitting up the bias and simultaneously following multiple strategies given by the Erdős-Selfridge criterion to avoid ‘clustering’.

In our paper we extend the ideas behind their proof to obtain general Maker and Breaker Win Criteria and apply them to the Rado games. These General criteria also allow one to generalize the result of Bednarska and Łuczak to hypergraphs of higher uniformity.

Here I will use the stronger ingredient of a probabilistic Ramsey statement for Maker’s part and give an outline of the proof for Breaker’s strategy.
**Theorem (Schacht; Conlon and Gowers ’10)**

For all positive and density regular $A \in \mathbb{Z}^{r \times m}$ and $\varepsilon > 0$ there exist $c, C$:

$$
\lim_{n \to \infty} \mathbb{P} \left( [n]_p \to \varepsilon A \right) = \begin{cases} 
0 & \text{if } p(n) \leq c n^{-1/m_1(A)}, \\
1 & \text{if } p(n) \geq C n^{-1/m_1(A)}. 
\end{cases}
$$
Theorem (Schacht; Conlon and Gowers ’10)

For all positive and density regular $A \in \mathbb{Z}^{r \times m}$ and $\varepsilon > 0$ there exist $c, C$:

$$\lim_{n \to \infty} \mathbb{P} ([n]_p \rightarrow \varepsilon A) = \begin{cases} 
0 & \text{if } p(n) \leq c n^{-1/m_1(A)}, \\
1 & \text{if } p(n) \geq C n^{-1/m_1(A)}. 
\end{cases}$$

Definition

Given $A \in \mathbb{Z}^{r \times m}$ let $\text{ex}(n, A)$ be the cardinality of the largest solution-free subset of $[n]$ and define $\pi(A) = \lim_{n \to \infty} \text{ex}(n, A)/n$. 

Maker’s Strategy: playing randomly
INTRODUCTION

Theorem (Schacht; Conlon and Gowers ’10)
For all positive and density regular \( A \in \mathbb{Z}^{r \times m} \) and \( \varepsilon > 0 \) there exist \( c, C \):

\[
\lim_{n \to \infty} P([n]_p \to \varepsilon \ A) = \begin{cases} 
0 & \text{if } p(n) \leq c n^{-1/m_1(A)}, \\
1 & \text{if } p(n) \geq C n^{-1/m_1(A)}. 
\end{cases}
\]

Definition
Given \( A \in \mathbb{Z}^{r \times m} \) let \( \text{ex}(n, A) \) be the cardinality of the largest solution-free subset of \([n]\) and define \( \pi(A) = \lim_{n \to \infty} \text{ex}(n, A)/n \).
Every positive and abundant matrix satisfies \( \pi(A) < 1 \).
Theorem (Schacht; Conlon and Gowers ’10)

For all positive and **density regular** $A \in \mathbb{Z}^{r \times m}$ and $\varepsilon > 0$ there exist $c, C$:

$$
\lim_{n \to \infty} \mathbb{P} ([n]_p \to \varepsilon A) = \begin{cases} 
0 & \text{if } p(n) \leq c n^{-1/m_1(A)}, \\
1 & \text{if } p(n) \geq C n^{-1/m_1(A)}. 
\end{cases}
$$

Definition

Given $A \in \mathbb{Z}^{r \times m}$ let $\text{ex}(n, A)$ be the cardinality of the largest solution-free subset of $[n]$ and define $\pi(A) = \lim_{n \to \infty} \text{ex}(n, A)/n$. Every positive and abundant matrix satisfies $\pi(A) < 1$.

Theorem (Hancock, Staden and Treglown ’17+; S. ’17+)

For every positive and **abundant** matrix $A \in \mathbb{Z}^{r \times m}$ and $\varepsilon > \pi(A)$ there exist constants $c(A, \varepsilon), C(A, \varepsilon) > 0$ such that

$$
\lim_{n \to \infty} \mathbb{P} ([n]_p \to \varepsilon A) = \begin{cases} 
0 & \text{if } p(n) \leq c n^{-1/m_1(A)}, \\
1 & \text{if } p(n) \geq C n^{-1/m_1(A)}. 
\end{cases}
$$
Proof.
Let an arbitrary strategy for Breaker be fixed.

**Proof.**
Let an arbitrary strategy for Breaker be fixed.
Proof.
Let an arbitrary strategy for Breaker be fixed.

1. Each round, Maker makes his move uniformly at random from among all elements of \([n]\) that he hasn’t previously picked.
Proof.
Let an arbitrary strategy for Breaker be fixed.

1. Each round, Maker makes his move uniformly at random from among all elements of $[n]$ that he hasn’t previously picked. If this element was already occupied by Breaker, he forfeits this move and we call it a failure.
Proof.

Let an arbitrary strategy for Breaker be fixed.

1. Each round, Maker makes his move uniformly at random from among all elements of \([n]\) that he hasn’t previously picked. If this element was already occupied by Breaker, he forfeits this move and we call it a \textit{failure}.

2. Pick an arbitrary \(\varepsilon > \pi(A)\) and let \(C = C(A, \varepsilon)\) be as given by the previous theorem. Set \(\delta = (1 - \varepsilon)/2\) and let \(q < \delta/(2C) n^{1/m_1(A)}\).
Proof.

Let an arbitrary strategy for Breaker be fixed.

1. Each round, Maker makes his move uniformly at random from among all elements of $[n]$ that he hasn’t previously picked. If this element was already occupied by Breaker, he forfeits this move and we call it a failure.

2. Pick an arbitrary $\varepsilon > \pi(A)$ and let $C = C(A, \varepsilon)$ be as given by the previous theorem. Set $\delta = (1 - \varepsilon)/2$ and let $q < \delta/(2C) n^{1/m_1(A)}$.

3. Stop after $M = \delta \lfloor n/(q + 1) \rfloor$ rounds so that Maker’s picks resemble a random graph $[n]_M$ where $M \geq 2C n^{1-1/m_1(A)}$. 

Maker’s Strategy: *playing randomly*
Proof.
Let an arbitrary strategy for Breaker be fixed.

1. Each round, Maker makes his move uniformly at random from among all elements of \([n]\) that he hasn’t previously picked. If this element was already occupied by Breaker, he forfeits this move and we call it a failure.

2. Pick an arbitrary \(\varepsilon > \pi(A)\) and let \(C = C(A, \varepsilon)\) be as given by the previous theorem. Set \(\delta = (1 - \varepsilon)/2\) and let \(q < \delta/(2C) n^{1/m_1(A)}\).

3. Stop after \(M = \delta \lfloor n/(q+1) \rfloor\) rounds so that Maker’s picks resemble a random graph \([n]_M\) where \(M \geq 2C n^{1-1/m_1(A)}\).

4. We have \(\mathbb{P}(\text{Maker’s } i\text{th move is a failure}) \leq \delta\), so by Markov’s inequality w.h.p. at least an \(\varepsilon\) fraction of his picks weren’t failures.
**Proof.**

Let an arbitrary strategy for Breaker be fixed.

1. Each round, Maker makes his move uniformly at random from among all elements of \([n]\) that he hasn’t previously picked. If this element was already occupied by Breaker, he forfeits this move and we call it a failure.

2. Pick an arbitrary \(\varepsilon > \pi(A)\) and let \(C = C(A, \varepsilon)\) be as given by the previous theorem. Set \(\delta = (1 - \varepsilon)/2\) and let \(q < \delta/(2C) n^{1/m_1(A)}\).

3. Stop after \(M = \delta \lceil n/(q+1) \rceil\) rounds so that Maker’s picks resemble a random graph \([n]_M\) where \(M \geq 2C n^{1-1/m_1(A)}\).

4. We have \(\mathbb{P}(\text{Maker’s } i\text{th move is a failure}) \leq \delta\), so by Markov’s inequality w.h.p. at least an \(\varepsilon\) fraction of his picks weren’t failures.

5. By the previous result, Maker’s random response succeeds a.a.s. so that there must exist a deterministic winning strategy. \(\square\)
Breaker’s Strategy: *avoiding clustering*

We need to aim at blocking some dominating substructure.
Breaker’s Strategy: *avoiding clustering*

We need to aim at blocking some dominating substructure. Let $Q_1 \subseteq [m]$ be a set of column indices satisfying $|Q_1| \geq 2$ such that

$$\frac{|Q_1| - 1}{|Q_1| - r_{Q_1} - 1} = m_1(A)$$  \hspace{1cm} (4)

and $|Q_1|$ is as small as possible.
Breaker’s Strategy: *avoiding clustering*

We need to aim at blocking some dominating substructure. Let $Q_1 \subseteq [m]$ be a set of column indices satisfying $|Q_1| \geq 2$ such that

$$\frac{|Q_1| - 1}{|Q_1| - r_{Q_1} - 1} = m_1(A)$$

(4)

and $|Q_1|$ is as small as possible. Consider the matrix $A[Q_1]$:

$$A \equiv \begin{pmatrix} A[Q_1] & 0 \\ 0 & 0 \end{pmatrix} \left[ \begin{array}{c} \text{rk}(A) - r_{Q_1} \\ r_{Q_1} \\ r - \text{rk}(A) \end{array} \right]$$

(5)
Breaker’s Strategy: *avoiding clustering*

We need to aim at blocking some dominating substructure. Let $Q_1 \subseteq [m]$ be a set of column indices satisfying $|Q_1| \geq 2$ such that

$$\frac{|Q_1| - 1}{|Q_1| - r_{Q_1} - 1} = m_1(A)$$

and $|Q_1|$ is as small as possible. Consider the matrix $A[Q_1]$:

$$A \cong \begin{pmatrix} \begin{array}{ccc} \hline A[Q_1] & \vline & 0 \\ \hline \end{array} & \begin{array}{c} \vline \\ \hline \end{array} & \begin{array}{c} \vline \\ \hline \end{array} \begin{array}{c} \rk(A) - r_{Q_1} \\ r_{Q_1} \\ r - \rk(A) \end{array} \end{pmatrix}$$

$A[Q_1]$ is positive and abundant.
Breaker’s Strategy: *avoiding clustering*

We need to aim at blocking some dominating substructure. Let \( Q_1 \subseteq [m] \) be a set of column indices satisfying \(|Q_1| \geq 2\) such that

\[
(\|Q_1\| - 1)/(\|Q_1\| - r_{Q_1} - 1) = m_1(A)
\]  

(4)

and \(|Q_1|\) is as small as possible. Consider the matrix \( A[Q_1] \):

\[
A \equiv \begin{pmatrix}
A[Q_1] & 0 \\
0 & 0
\end{pmatrix}
\begin{bmatrix}
\text{rk}(A) - r_{Q_1} \\
r_{Q_1} \\
r - \text{rk}(A)
\end{bmatrix}
\]  

(5)

\( A[Q_1] \) is positive and abundant. Furthermore, blocking solutions to \( A[Q_1] \) also blocks solutions to \( A \):

**Lemma**

*Let \( T \subseteq \mathbb{N} \) and \( Q_1 \subseteq [m] \) as above. If there does not exist a solution to \( A[Q_1] \cdot x^T = 0^T \) in \( T \) then there also does not exist a solution to \( A \cdot x^T = 0^T \).*
Breaker’s Strategy: *avoiding clustering*

**Remark (Strategy Splitting)**

*If Breaker has a winning strategy in $\mathcal{H}_1$ and $\mathcal{H}_2$ with a bias of $q_1$ and $q_2$ respectively, then he has a winning strategy in $\mathcal{H}_1 \cup \mathcal{H}_2$ with a bias of $q_1 + q_2$.***
Breaker’s Strategy: avoiding clustering

Remark (Strategy Splitting)
If Breaker has a winning strategy in $\mathcal{H}_1$ and $\mathcal{H}_2$ with a bias of $q_1$ and $q_2$ respectively, then he has a winning strategy in $\mathcal{H}_1 \cup \mathcal{H}_2$ with a bias of $q_1 + q_2$.

Definition
Let $\mathcal{H}_n$ be the hypergraph of all proper solutions to $A \cdot x = 0$ in $[n]$. 

Breaker’s Strategy: *avoiding clustering*

Remark (Strategy Splitting)

If Breaker has a winning strategy in $\mathcal{H}_1$ and $\mathcal{H}_2$ with a bias of $q_1$ and $q_2$ respectively, then he has a winning strategy in $\mathcal{H}_1 \cup \mathcal{H}_2$ with a bias of $q_1 + q_2$.

Definition

Let $\mathcal{H}_n$ be the hypergraph of all proper solutions to $A \cdot x = 0$ in $[n]$.

1. A *t-cluster* is any family of distinct edges $\{H_1, \ldots, H_t\} \subset \mathcal{H}_n$ satisfying $|\bigcap_{i=1}^{t} H_i| \geq 1$, 


Breaker’s Strategy: avoiding clustering

Remark (Strategy Splitting)

If Breaker has a winning strategy in $\mathcal{H}_1$ and $\mathcal{H}_2$ with a bias of $q_1$ and $q_2$ respectively, then he has a winning strategy in $\mathcal{H}_1 \cup \mathcal{H}_2$ with a bias of $q_1 + q_2$.

Definition

Let $\mathcal{H}_n$ be the hypergraph of all proper solutions to $A \cdot x = 0$ in $[n]$.

1. A $t$-cluster is any family of distinct edges $\{H_1, \ldots, H_t\} \subset \mathcal{H}_n$ satisfying $|\bigcap_{i=1}^{t} H_i| \geq 1$,

2. an almost complete solution is a tuple $(H^\circ, h)$ consisting of a set $H^\circ \subseteq V(\mathcal{H}_n)$ as well as $h \notin H^\circ$ so that $H = H^\circ \cup \{h\} \in \mathcal{H}_n$,
Breaker’s Strategy: *avoiding clustering*

Remark (Strategy Splitting)

*If Breaker has a winning strategy in $\mathcal{H}_1$ and $\mathcal{H}_2$ with a bias of $q_1$ and $q_2$ respectively, then he has a winning strategy in $\mathcal{H}_1 \cup \mathcal{H}_2$ with a bias of $q_1 + q_2$.***

Definition

Let $\mathcal{H}_n$ be the hypergraph of all proper solutions to $A \cdot x = 0$ in $[n]$.

1. A *t-cluster* is any family of distinct edges $\{H_1, \ldots, H_t\} \subset \mathcal{H}_n$ satisfying $|\bigcap_{i=1}^t H_i| \geq 1$,

2. an *almost complete solution* is a tuple $(H^\circ, h)$ consisting of a set $H^\circ \subseteq V(\mathcal{H}_n)$ as well as $h \notin H^\circ$ so that $H = H^\circ \cup \{h\} \in \mathcal{H}_n$,

3. a *t-fan* is a family of distinct almost complete solutions $\{(H_1^\circ, h_1), \ldots, (H_t^\circ, h_t)\}$ in $\mathcal{H}_n$ satisfying $|\bigcap_{i=1}^t H_i^\circ| \geq 1$. 

*An almost complete solution $(H^\circ, h)$ is dangerous if $H^\circ$ has been covered by Maker and $h$ has not yet been picked by either player. A fan is dangerous if its respective almost complete solutions are.*
Breaker’s Strategy: avoiding clustering

Remark (Strategy Splitting)

If Breaker has a winning strategy in $H_1$ and $H_2$ with a bias of $q_1$ and $q_2$ respectively, then he has a winning strategy in $H_1 \cup H_2$ with a bias of $q_1 + q_2$.

Definition

Let $\mathcal{H}_n$ be the hypergraph of all proper solutions to $A \cdot x = 0$ in $[n]$.

1. A $t$-cluster is any family of distinct edges $\{H_1, \ldots, H_t\} \subset \mathcal{H}_n$ satisfying $|\bigcap_{i=1}^t H_i| \geq 1$.
2. An almost complete solution is a tuple $(\mathcal{H}^\circ, h)$ consisting of a set $\mathcal{H}^\circ \subseteq V(\mathcal{H}_n)$ as well as $h \notin \mathcal{H}^\circ$ so that $H = \mathcal{H}^\circ \cup \{h\} \in \mathcal{H}_n$.
3. A $t$-fan is a family of distinct almost complete solutions $\{(\mathcal{H}^\circ_1, h_1), \ldots, (\mathcal{H}^\circ_t, h_t)\}$ in $\mathcal{H}_n$ satisfying $|\bigcap_{i=1}^t \mathcal{H}^\circ_i| \geq 1$.

An almost complete solution $(\mathcal{H}^\circ, h)$ is dangerous if $\mathcal{H}^\circ$ has been covered by Maker and $h$ has not yet been picked by either player.
Breaker’s Strategy: *avoiding clustering*

**Remark (Strategy Splitting)**

If Breaker has a winning strategy in $\mathcal{H}_1$ and $\mathcal{H}_2$ with a bias of $q_1$ and $q_2$ respectively, then he has a winning strategy in $\mathcal{H}_1 \cup \mathcal{H}_2$ with a bias of $q_1 + q_2$.

**Definition**

Let $\mathcal{H}_n$ be the hypergraph of all proper solutions to $A \cdot x = 0$ in $[n]$.

1. A *$t$-cluster* is any family of distinct edges $\{H_1, \ldots, H_t\} \subseteq \mathcal{H}_n$ satisfying $|\bigcap_{i=1}^t H_i| \geq 1$,

2. an *almost complete solution* is a tuple $(H^\circ, h)$ consisting of a set $H^\circ \subseteq V(\mathcal{H}_n)$ as well as $h \not\in H^\circ$ so that $H = H^\circ \cup \{h\} \in \mathcal{H}_n$,

3. a *$t$-fan* is a family of distinct almost complete solutions $\{(H^\circ_1, h_1), \ldots, (H^\circ_t, h_t)\}$ in $\mathcal{H}_n$ satisfying $|\bigcap_{i=1}^t H^\circ_i| \geq 1$.

An almost complete solution $(H^\circ, h)$ is *dangerous* if $H^\circ$ has been covered by Maker and $h$ has not yet been picked by either player. A fan is dangerous if its respective almost complete solutions are.
Proposition

For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ Breaker wins the associated Rado game with a bias of $q \gg n^{1/m_1(A)}$. 
Proposition

For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ Breaker wins the associated Rado game with a bias of $q \gg n^{1/m_1(A)}$.

Proof.

1. Using the Erdős-Selfridge criterion, Breaker has a strategy that avoids $t$-clusters using some fraction the bias $q' = q/(t + 1) - 1$ where $t = t(A) \in \mathbb{N}$ is a large constant.
Breaker’s Strategy: avoid clustering

Proposition

For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ Breaker wins the associated Rado game with a bias of $q \gg n^{1/m_1(A)}$.

Proof.

1. Using the Erdős-Selfridge criterion, Breaker has a strategy that avoids $t$-clusters using some fraction the bias $q' = q/(t + 1) - 1$ where $t = t(A) \in \mathbb{N}$ is a large constant.

2. The same strategy must also avoid dangerous $t (q' + 1)$-fans.
Breaker’s Strategy: avoid clustering

Proposition

For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ Breaker wins the associated Rado game with a bias of $q \gg n^{1/m_1(A)}$.

Proof.

1. Using the Erdős-Selfridge criterion, Breaker has a strategy that avoids $t$-clusters using some fraction the bias $q' = q/(t + 1) - 1$ where $t = t(A) \in \mathbb{N}$ is a large constant.

2. The same strategy must also avoid dangerous $t(q' + 1)$-fans.

3. Using the remaining $q - q' \geq t(q' + 1)$ moves it follows inductively that each round Breaker can neutralise every dangerous almost complete solution and hence win.
Breaker’s Strategy: avoid clustering

Proposition

For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ Breaker wins the associated Rado game with a bias of $q \gg n^{1/m_1(A)}$.

Proof.

1. Using the Erdős-Selfridge criterion, Breaker has a strategy that avoids $t$-clusters using some fraction the bias $q' = q/(t + 1) - 1$ where $t = t(A) \in \mathbb{N}$ is a large constant.

2. The same strategy must also avoid dangerous $t (q' + 1)$-fans.

3. Using the remaining $q - q' \geq t (q' + 1)$ moves it follows inductively that each round Breaker can neutralise every dangerous almost complete solution and hence win.

To get to the correct threshold, one combines a strategy as above aimed at structures intersecting in at least 2 points with another application of Erdős-Selfridge aimed at structures intersecting in exactly 1 point.
Breaker’s Strategy: avoid clustering

Proposition

For all positive and abundant matrices \( A \in \mathbb{Z}^{r \times m} \) Breaker wins the associated Rado game with a bias of \( q \gg n^{1/m_1(A)} \).

Proof.

1. Using the Erdős-Selfridge criterion, Breaker has a strategy that avoids \( t \)-clusters using some fraction the bias \( q' = q / (t + 1) - 1 \) where \( t = t(A) \in \mathbb{N} \) is a large constant.

2. The same strategy must also avoid dangerous \( t \left( q' + 1 \right) \)-fans.

3. Using the remaining \( q - q' \geq t \left( q' + 1 \right) \) moves it follows inductively that each round Breaker can neutralise every dangerous almost complete solution and hence win.

To get to the correct threshold, one combines a strategy as above aimed at structures intersecting in at least 2 points with another application of Erdős-Selfridge aimed at structures intersecting in exactly 1 point. One then combines the two results through an auxiliary lemma.
Open Question

Q1. Can one obtain good bounds for the constants?
Open Question

**Q1.** Can one obtain good bounds for the constants?

**Conjecture**

For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ there exists a constant $c = c(A)$ such that for $\epsilon > 0$ and $n$ large enough Breaker has a winning strategy with a bias of $q > (c + \epsilon) n^{1/m_1(A)}$ and Maker has a winning strategy if $q < (c - \epsilon) n^{1/m_1(A)}$. 
Open Question

**Q1.** Can one obtain good bounds for the constants?

**Conjecture**

For all positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ there exists a constant $c = c(A)$ such that for $\varepsilon > 0$ and $n$ large enough Breaker has a winning strategy with a bias of $q > (c + \varepsilon) n^{1/m_1(A)}$ and Maker has a winning strategy if $q < (c - \varepsilon) n^{1/m_1(A)}$.

**Q2.** Can one formulate an explicit winning strategy for Maker?
Thank you for your attention!