On a problem of Sárközy and Sós for multivariate linear forms

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Some general Motivation: Gauss’ Circle Problem

**Q**: How many integer lattice points are in a circle with radius $r$ centred at the origin?
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Theorem (Huxley 2003)

We have \( E(r) = O(r^{131/208}) \).
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A: $\#\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 \leq r^2\} = \pi r^2 + E(r)$

Theorem (Huxley 2003)

*We have $E(r) = O(r^{131/208})$.*

Theorem (Hardy 1915; Landau 1915)

*We cannot have $E(r) = o(r^{1/2} \log(r)^{1/4})$.***
Additive representation functions

Definition
For any infinite set $\mathcal{A} \subseteq \mathbb{N}_0$ and $n \in \mathbb{N}_0$, let

$$r_\mathcal{A}(n) = \#\{(a_1, a_2) \in \mathcal{A}^2 : a_1 + a_2 = n\}. \quad (1)$$
### Additive representation functions

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**Remark**
Trivially $r_{\mathcal{A}}(n)$ is odd if $n = 2a$ for some $a \in \mathcal{A}$ and even otherwise.
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**Theorem (Erdős and Fuchs 1956)**
For any infinite $\mathcal{A} \subseteq \mathbb{N}$ and $c > 0$ we **cannot** have

$$\sum_{n=1}^{N} r_\mathcal{A}(n) = cN + o(N^{1/4} \log N^{-1/2}).$$  \hfill \(2\)
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For any infinite set $A \subseteq \mathbb{N}_0$ and $n \in \mathbb{N}_0$, let

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Corollary
Considering the case where $A = \{m^2 : m \in \mathbb{N}\}$, $c = \pi/4$ and $N = r^2 - 4r/\pi$, it follows that we cannot have $E(r) = o(r^{1/2} \log(r)^{-1/2})$. 
**Additive representation functions**

**Sárközy and Sós ’97:** For which \( k_1, \ldots, k_d \in \mathbb{N} \) does there exist an infinite set \( A \subseteq \mathbb{N}_0 \) and \( n_0 \geq 0 \) such that

\[
r_A(n; k_1, \ldots, k_d) = \# \{(a_1, \ldots, a_d) \in A^d : k_1 a_1 + \cdots + k_d a_d = n \}
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is constant for \( n \geq n_0 \)?
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*We already observed that \(r_A(n; 1, 1)\) cannot become constant.*
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**Theorem (Moser 1962)**

*For any $k \geq 2$ there exists $A \subseteq \mathbb{N}_0$ such that $r_A(n; 1, k, k^2, \ldots, k^{d-1}) = 1.$*
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*For any $k_1, k_2 \geq 2$ and $A \subseteq \mathbb{N}_0$, $r_A(n; k_1, k_2)$ cannot become constant.*
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Theorem (Rué and S. 2018+)
If there are pairwise co-prime integers $q_1, \ldots, q_m \geq 2$ such that

$$k_i = q_1^{b(i,1)} \cdots q_m^{b(i,m)} \geq 2$$ (3)

where $b(i, j) \in \{0, 1\}$, then $r_A(n; k_1, \ldots, k_d)$ cannot become constant for any infinite $A \subseteq \mathbb{N}_0$. 

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where \( b(i,j) \in \{0,1\} \), then \( r_A(n; k_1, \ldots, k_d) \) cannot become constant for any infinite \( A \subseteq \mathbb{N}_0 \). This includes the case of pairwise co-prime \( k_1, \ldots, k_d \geq 2 \).
The proof of Moser’s result

**Theorem (Moser 1962)**

*For any* $k \geq 2$ *there exists* $A \subseteq \mathbb{N}_0$ *such that* $r_A(n; 1, k) = 1$ *for all* $n \geq 0$. 
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For any $k \geq 2$ there exists $A \subseteq \mathbb{N}_0$ such that $r_A(n; 1, k) = 1$ for all $n \geq 0$.

Proof.

The generating function of $A$ is $f_A(z) = \sum_{a \in A} z^a$. 

Writing $f_A(z) = \left(1 - z\right)^{-1} f^{-1}_A(z^k)$ and repeatedly substituting, we get $f_A(z) = \prod_{j=0}^{\infty} \left(1 + z^{k^2} j + z^{2k^2} j + \cdots + z^{(k-1)k^2} j \right)$. This is the representation function of the set of all integers whose $k^2$-ary representation has only digits strictly smaller than $k$. 


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$$f_A(z) f_A(z^k) = \sum_{(a,a') \in A^2} z^{a+ka'} = \sum_{n=0}^{\infty} r(n; 1, k) z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}. \quad (4)$$
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*For any* \( k \geq 2 \) *there exists* \( A \subseteq \mathbb{N}_0 \) *such that* \( r_A(n; 1, k) = 1 \) *for all* \( n \geq 0 \).

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Writing \( f_A(z) = (1 - z)^{-1} f_A^{-1}(z^k) \) and repeatedly substituting,
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$$f_A(z) = \prod_{j=0}^\infty \left( 1 + z^{(k^2)^j} + z^2(k^2)^j + \cdots + z^{(k-1)(k^2)^j} \right).$$
The proof of Moser’s result

Theorem (Moser 1962)
For any \( k \geq 2 \) there exists \( \mathcal{A} \subseteq \mathbb{N}_0 \) such that \( r_{\mathcal{A}}(n; 1, k) = 1 \) for all \( n \geq 0 \).

Proof.
The generating function of \( \mathcal{A} \) is \( f_{\mathcal{A}}(z) = \sum_{a \in \mathcal{A}} z^a \). We have

\[
\begin{align*}
f_{\mathcal{A}}(z)f_{\mathcal{A}}(z^k) &= \sum_{(a,a') \in \mathcal{A}^2} z^{a+ka'} = \sum_{n=0}^{\infty} r(n; 1, k) z^n \\
&= \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.
\end{align*}
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Writing \( f_{\mathcal{A}}(z) = (1 - z)^{-1}f_{\mathcal{A}}^{-1}(z^k) \) and repeatedly substituting, we get

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This is the representation function of the set of all integers whose \( k^2 \)-ary representation has only digits strictly smaller than \( k \). \( \square \)
Proof Outline of Main Result

Proof Outline.
From here on we will abbreviate $r_A(n) = r_A(n; k_1, \ldots, k_d)$. 

If $r_A(n)$ becomes constant, that is there exist $c > 0$ and $n_0 \geq 0$ such that $r_A(n) = c$ for $n \geq n_0 \geq 0$, then 

$$
\sum_{n=0}^{\infty} r_A(n) z^n = n_0 - 1 \sum_{n=0}^{\infty} r_A(n) z^n + \sum_{n=n_0}^{\infty} c z^n = Q(z) + c z^{n_0} 1 - z = P(z) 1 - z
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where $Q \in \mathbb{N}[z]$ and $P \in \mathbb{Z}[z]$ are polynomials and $P(1) \neq 0$. 

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Generalising the previous approach, we have

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 f_{\mathcal{A}}(z^{k_1}) \cdots f_{\mathcal{A}}(z^{k_d}) = \sum_{(a_1, \ldots, a_d) \in \mathcal{A}^d} z^{k_1a_1 + \cdots + k_da_d} = \sum_{n=0}^{\infty} r_{\mathcal{A}}(n) z^n. \tag{5}
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$$\sum_{n=0}^{\infty} r_A(n) z^n = \sum_{n=0}^{n_0-1} r_A(n) z^n + \sum_{n=n_0}^{\infty} c z^n = Q(z) + c \frac{z^{n_0}}{1-z} = \frac{P(z)}{1-z} \quad (6)$$
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where \( Q \in \mathbb{N}_0[z] \) and \( P \in \mathbb{Z}[z] \) are polynomials and \( P(1) \neq 0 \).
Proof Outline.

From here on we will abbreviate $r_{A}(n) = r_{A}(n; k_1, \ldots, k_d)$. Generalising the previous approach, we have

$$f_{A}(z^{k_1}) \cdots f_{A}(z^{k_d}) = \sum_{(a_1, \ldots, a_d) \in \mathcal{A}^d} z^{a_1k_1 + \cdots + a_dk_d} = \sum_{n=0}^{\infty} r_{A}(n) z^n. \quad (5)$$

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where $Q \in \mathbb{N}_0[z]$ and $P \in \mathbb{Z}[z]$ are polynomials and $P(1) \neq 0$. We have

$$f_{A}(z^{k_1}) \cdots f_{A}(z^{k_d}) = \frac{P(z)}{1-z}. \quad (7)$$
The *cyclotomic polynomial* of order $n$ is given by

$$
\Phi_n(z) = \prod_{\xi \in \phi_n} (z - \xi)
$$

where $\phi_n = \{\xi \in \mathbb{C} : \xi^k = 1 \text{ iff } k \equiv 0 \pmod{n}\}$. 
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The *cyclotomic polynomial* of order $n$ is given by

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Introducing cyclotomic polynomials
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Recall that $f_\lambda (z^{k_1}) \cdots f_\lambda (z^{k_d}) = P(z)/(1 - z)$ for some $P(z) \in \mathbb{Z}[z]$ satisfying $P(1) \neq 0$. 

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Recall that $f_A(z^{k_1}) \cdots f_A(z^{k_d}) = P(z)/(1 - z)$ for some $P(z) \in \mathbb{Z}[z]$ satisfying $P(1) \neq 0$. Now, for any $n$ there exists a unique $s_n \in \mathbb{N}_0$ s.t.

$$P_n(z) = P(z) \Phi_n^{-s_n}(z) \quad (9)$$

satisfies $P_n(z) \in \mathbb{Z}[z]$ as well as $P_n(\xi) \neq 0$ for any $\xi \in \phi_n$. 


Factoring out the generating function

If \( r_A(n) \) becomes constant for some \( A \subseteq \mathbb{N}_0 \), then

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Factoring out the generating function

If \( r_{\mathcal{A}}(n) \) becomes constant for some \( \mathcal{A} \subseteq \mathbb{N}_0 \), then

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(9)

Proposition

For any \( (j_1, \ldots, j_d) \in \mathbb{N}_0^d \) there exist \( r_j \) satisfying

\[
\lim_{\omega \to 1} f(\omega \xi) \cdot \Phi^{-r_j}_{k_1^{j_1} \cdots k_d^{j_d}}(\omega \xi) \notin \{0, \pm \infty\}
\]

(10)

for any \( \xi \in \phi_{k_1^{j_1} \cdots k_d^{j_d}}. \)
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If \( r_A(n) \) becomes constant for some \( A \subseteq \mathbb{N}_0 \), then

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for any \( \xi \in \phi_{k_1^{j_1} \cdots k_d^{j_d}} \). These exponents satisfy \( r_0 = -1/d \).
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If \( r_{\mathcal{A}}(n) \) becomes constant for some \( \mathcal{A} \subseteq \mathbb{N}_0 \), then

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f_{\mathcal{A}}(z^{k_1}) \cdots f_{\mathcal{A}}(z^{k_d}) = \frac{P(z)}{1 - z} \quad \text{where} \quad P(1) \neq 0 \tag{7}
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\]

**Proposition**

For any \( (j_1, \ldots, j_d) \in \mathbb{N}_0^d \) there exist \( r_j \) satisfying

\[
\lim_{\omega \to 1} f(\omega \xi) \cdot \Phi_{k_1 \cdots k_d}^{-r_j}(\omega \xi) \notin \{0, \pm \infty\} \tag{10}
\]

for any \( \xi \in \phi_{k_1 \cdots k_d} \). These exponents satisfy \( r_0 = -1/d \) and

\[
r_{j_1 \oplus b(1,1), \ldots, j_d \oplus b(d,1)} + \cdots + r_{j_1 \oplus b(1,m), \ldots, j_d \oplus b(d,m)} = ds_j \tag{11}
\]

for all \( j \in \mathbb{N}_0^m \setminus \{0\} \) where \( a \oplus b = \max(a - b, 0) \) and \( s_j = s_{k_1 \cdots k_d} \).
Finding a contradiction in the exponents

Consider the case of Rué and Cilleruelo, that is we have $d = 2$. 
Finding a contradiction in the exponents

Consider the case of Rué and Cilleruelo, that is we have $d = 2$. The proposition gives the existence of $\{r_j : j \in \mathbb{N}_0^2\}$ satisfying

(i) $r_{(0,0)} = -1/2$,

(ii) $r_{(j+1,0)} = s_{(j+1,0)} - r_{(j,0)}$,

(iii) $r_{(0,j+1)} = s_{(0,j+1)} - r_{(0,j)}$ and

(iv) $r_{(j_1+1,j_2)} + r_{(j_1,j_2+1)} = s_{(j_1+1,j_2+1)}$. 

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Inductively, as \( s_* \in \mathbb{N}_0 \), we have \( r_* \notin \mathbb{Z} \) and therefore \( r_* \neq 0 \) due to (i).
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Inductively, as $s_* \in \mathbb{N}_0$, we have $r_* \notin \mathbb{Z}$ and therefore $r_* \neq 0$ due to (i). As $P$ is a polynomial there exists $\ell_0$ such that $s_{j_1,j_2} = 0$ if $j_1 + j_2 \geq \ell_0$. 
Finding a contradiction in the exponents

Consider the case of Rué and Cilleruelo, that is we have \( d = 2 \). The proposition gives the existence of \( \{r_j : j \in \mathbb{N}_0^2\} \) satisfying

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As \( P \) is a polynomial there exists \( \ell_0 \) such that \( s_{j_1,j_2} = 0 \) if \( j_1 + j_2 \geq \ell_0 \).

Assume w.l.o.g. that \( \ell_0 \) is odd.
Finding a contradiction in the exponents

Consider the case of Rué and Cilleruelo, that is we have $d = 2$. The proposition gives the existence of $\{r_j : j \in \mathbb{N}_0^2\}$ satisfying

(i) $r_{(0,0)} = -1/2$,
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Inductively, as $s_* \in \mathbb{N}_0$, we have $r_* \notin \mathbb{Z}$ and therefore $r_* \neq 0$ due to (i). As $P$ is a polynomial there exists $\ell_0$ such that $s_{j_1,j_2} = 0$ if $j_1 + j_2 \geq \ell_0$. Assume w.l.o.g. that $\ell_0$ is odd. Now

- $r_{(\ell_0+1,0)} = -r_{(\ell_0,0)}$ due to (ii),
Finding a contradiction in the exponents

Consider the case of Rué and Cilleruelo, that is we have $d = 2$. The proposition gives the existence of $\{r_j : j \in \mathbb{N}_0^2\}$ satisfying

(i) $r_{(0,0)} = -1/2$,

(ii) $r_{(j+1,0)} = s_{(j+1,0)} - r_{(j,0)}$,

(iii) $r_{(0,j+1)} = s_{(0,j+1)} - r_{(0,j)}$ and

(iv) $r_{(j_1+1,j_2)} + r_{(j_1,j_2+1)} = s_{(j_1+1,j_2+1)}$.

Inductively, as $s_\ast \in \mathbb{N}_0$, we have $r_\ast \notin \mathbb{Z}$ and therefore $r_\ast \neq 0$ due to (i). As $P$ is a polynomial there exists $\ell_0$ such that $s_{j_1,j_2} = 0$ if $j_1 + j_2 \geq \ell_0$. Assume w.l.o.g. that $\ell_0$ is odd. Now

- $r_{(\ell_0+1,0)} = -r_{(\ell_0,0)}$ due to (ii),
- $r_{(0,\ell_0+1)} = -r_{(0,\ell_0)}$ due to (iii),
Finding a contradiction in the exponents

Consider the case of Rué and Cilleruelo, that is we have $d = 2$. The proposition gives the existence of \( \{r_j : j \in \mathbb{N}_0^2\} \) satisfying

\begin{align*}
(i) & \quad r_{(0,0)} = -1/2, \\
(ii) & \quad r_{(j+1,0)} = s_{(j+1,0)} - r_{(j,0)}, \\
(iii) & \quad r_{(0,j+1)} = s_{(0,j+1)} - r_{(0,j)} \text{ and} \\
(iv) & \quad r_{(j_1+1,j_2)} + r_{(j_1,j_2+1)} = s_{(j_1+1,j_2+1)}.
\end{align*}

Inductively, as $s_* \in \mathbb{N}_0$, we have $r_* \notin \mathbb{Z}$ and therefore $r_* \neq 0$ due to (i). As $P$ is a polynomial there exists $\ell_0$ such that $s_{j_1,j_2} = 0$ if $j_1 + j_2 \geq \ell_0$. Assume w.l.o.g. that $\ell_0$ is odd. Now

- $r_{(\ell_0+1,0)} = -r_{(\ell_0,0)}$ due to (ii),
- $r_{(0,\ell_0+1)} = -r_{(0,\ell_0)}$ due to (iii),
- $r_{(\ell_0,0)} = r_{(0,\ell_0)}$ and $r_{(\ell_0+1,0)} = -r_{(0,\ell_0+1)}$ due to (iv)
Finding a contradiction in the exponents

Consider the case of Rué and Cilleruelo, that is we have $d = 2$. The proposition gives the existence of $\{ r_j : j \in \mathbb{N}_0^2 \}$ satisfying

(i) $r_{(0,0)} = -1/2$,
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(iv) $r_{(j_1+1,j_2)} + r_{(j_1,j_2+1)} = s_{(j_1+1,j_2+1)}$.

Inductively, as $s_* \in \mathbb{N}_0$, we have $r_* \notin \mathbb{Z}$ and therefore $r_* \neq 0$ due to (i). As $P$ is a polynomial there exists $\ell_0$ such that $s_{j_1,j_2} = 0$ if $j_1 + j_2 \geq \ell_0$. Assume w.l.o.g. that $\ell_0$ is odd. Now

- $r_{(\ell_0+1,0)} = -r_{(\ell_0,0)}$ due to (ii),
- $r_{(0,\ell_0+1)} = -r_{(0,\ell_0)}$ due to (iii),
- $r_{(\ell_0,0)} = r_{(0,\ell_0)}$ and $r_{(\ell_0+1,0)} = -r_{(0,\ell_0+1)}$ due to (iv)

implying the contradiction $r_{(\ell_0,0)} = r_{(0,\ell_0)} = r_{(\ell_0+1,0)} = r_{(0,\ell_0+1)} = 0$. □
Remarks and Open Problems

Conjecture

The cases covered by Moser, that is $1, k, k^2, \ldots, k^{d-1}$, are the only ones for which $r_A(n)$ can become constant.

1. What about cases not covered by our result, e.g. $r_A(n; 2, 3, 4)$ or $r_A(1, 2, 6)$?

2. What about the unordered variant

$$R_A(n; k_1, \ldots, k_d) = \# \left\{ \{a_1, \ldots, a_d\} \in 2^A : k_1 a_1 + \cdots + k_d a_d = n \right\} ?$$

3. What about an Erdős-Fuchs-type result for $k_1 = 2$ and $k_2 = 3$?
Thank you for your attention!