On a problem of Sárközy and Sós for multivariate linear forms

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Discrete Mathematics Days

Sevilla, June 2018

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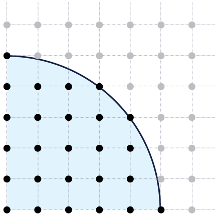
Some general Motivation: Gauss' Circle Problem

Q: How many integer lattice points are in a circle with radius r centred at the origin?



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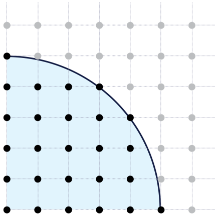

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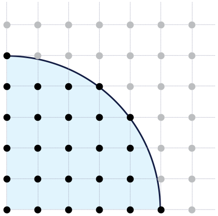

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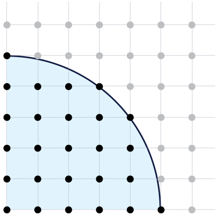

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We have $E(r) = O(r^{131/208})$.

Theorem (Hardy 1915; Landau 1915)

We **cannot** *have* $E(r) = o(r^{1/2} \log(r)^{1/4})$.

Additive representation functions

Definition

For any infinite set $\mathcal{A} \subseteq \mathbb{N}_0$ and $n \in \mathbb{N}_0$, let

$$r_{\mathcal{A}}(n) = \#\{(a_1, a_2) \in \mathcal{A}^2 : a_1 + a_2 = n\}. \tag{1}$$

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Trivially $r_{\mathcal{A}}(n)$ is odd if $n = 2a$ for some $a \in \mathcal{A}$ and even otherwise.

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Theorem (Erdős and Fuchs 1956)

For any infinite $\mathcal{A} \subseteq \mathbb{N}$ and $c > 0$ we **cannot** *have*

$$\sum_{n=1}^{N} r_{\mathcal{A}}(n) = cN + o\big(N^{1/4} \log N^{-1/2}\big). \tag{2}$$

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Corollary
Considering the case where $\mathcal{A} = \{m^2 : m \in \mathbb{N}\}$, $c = \pi/4$ and $N = r^2 - 4r/\pi$, it follows that we **cannot** *have $E(r) = o\left(r^{1/2} \log(r)^{-1/2}\right)$.*

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Sárközy and Sós '97: For which $k_1, \ldots, k_d \in \mathbb{N}$ does there exist an infinite set $\mathcal{A} \subseteq \mathbb{N}_0$ and $n_0 \geq 0$ such that

$$r_{\mathcal{A}}(n; k_1, \ldots, k_d) = \#\{(a_1, \ldots, a_d) \in \mathcal{A}^d : k_1 a_1 + \cdots + k_d a_d = n\}$$

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If there are pairwise co-prime integers $q_1, \ldots, q_m \geq 2$ *such that*

$$k_i = q_1^{b(i,1)} \cdots q_m^{b(i,m)} \geq 2 \tag{3}$$

where $b(i,j) \in \{0,1\}$, *then* $r_{\mathcal{A}}(n; k_1, \ldots, k_d)$ **cannot** *become constant for any infinite* $\mathcal{A} \subseteq \mathbb{N}_0$.

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where $b(i,j) \in \{0,1\}$, then $r_{\mathcal{A}}(n; k_1, \ldots, k_d)$ **cannot** *become constant for any infinite $\mathcal{A} \subseteq \mathbb{N}_0$. This includes the case of pairwise co-prime $k_1, \ldots, k_d \geq 2$.*

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This is the representation function of the set of all integers whose k^2-ary representation has only digits strictly smaller than k. □

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Introducing cyclotomic poylinomials

The *cyclotomic polynomial* of order n is given by

$$\Phi_n(z) = \prod_{\xi \in \phi_n} (z - \xi) \tag{8}$$

where $\phi_n = \{\xi \in \mathbb{C} : \xi^k = 1 \text{ iff } k \equiv 0 \mod n\}$.

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$$P_n(z) = P(z)\, \Phi_n^{-s_n}(z) \tag{9}$$

satisfies $P_n(z) \in \mathbb{Z}[z]$ as well as $P_n(\xi) \neq 0$ for any $\xi \in \phi_n$.

Factoring out the generating function

If $r_{\mathcal{A}}(n)$ becomes constant for some $\mathcal{A} \subseteq \mathbb{N}_0$, then

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$$\lim_{\omega \to 1} f(\omega\xi) \cdot \Phi^{-r_j}_{k_1^{j_1} \cdots k_d^{j_d}}(\omega\xi) \notin \{0, \pm\infty\} \tag{10}$$

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for any $\xi \in \phi_{k_1^{j_1} \cdots k_d^{j_d}}$. *These exponents satisfy* $r_{\mathbf{0}} = -1/d$

Factoring out the generating function

If $r_{\mathcal{A}}(n)$ becomes constant for some $\mathcal{A} \subseteq \mathbb{N}_0$, then

$$f_{\mathcal{A}}(z^{k_1}) \cdots f_{\mathcal{A}}(z^{k_d}) = \frac{P(z)}{1-z} \quad \text{where} \quad P(1) \neq 0 \tag{7}$$

and there exist $s_n \in \mathbb{N}_0$ such that

$$P_n(z) = P(z)\,\Phi_n^{-s_n}(z) \text{ satisfies } P_n(\xi) \neq 0 \text{ for } \xi \in \phi_n. \tag{9}$$

Proposition

For any $(j_1, \ldots, j_d) \in \mathbb{N}_0^d$ there exist $r_{\boldsymbol{j}}$ satisfying

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for any $\xi \in \phi_{k_1^{j_1} \cdots k_d^{j_d}}$. These exponents satisfy $r_{\mathbf{0}} = -1/d$ and

$$r_{(j_1 \ominus b(1,1), \ldots, j_d \ominus b(d,1))} + \cdots + r_{(j_1 \ominus b(1,m), \ldots, j_d \ominus b(d,m))} = d s_{\boldsymbol{j}} \tag{11}$$

for all $\boldsymbol{j} \in \mathbb{N}_0^m \setminus \{\mathbf{0}\}$ where $a \ominus b = \max(a-b, 0)$ and $s_{\boldsymbol{j}} = s_{k_1^{j_1} \cdots k_d^{j_d}}$.

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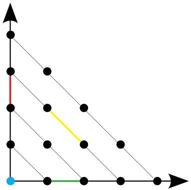

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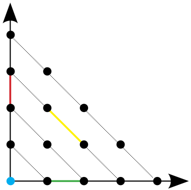

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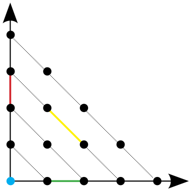

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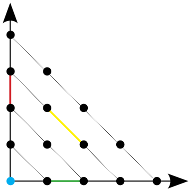

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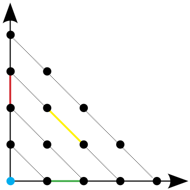

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- $r_{(\ell_0+1,0)} = -r_{(\ell_0,0)}$ due to (ii),

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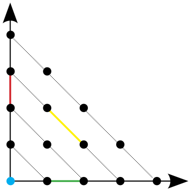

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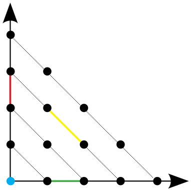

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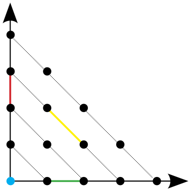

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implying the contradiction $r_{(\ell_0,0)} = r_{(0,\ell_0)} = r_{(\ell_0+1,0)} = r_{(0,\ell_0+1)} = 0$. □

Remarks and Open Problems

Conjecture

The cases covered by Moser, that is $1, k, k^2, \ldots, k^{d-1}$, are the only ones for which $r_{\mathcal{A}}(n)$ can become constant.

1. What about cases not covered by our result, e.g. $r_{\mathcal{A}}(n; 2, 3, 4)$ or $r_{\mathcal{A}}(1, 2, 6)$?
2. What about the unordered variant

$$R_{\mathcal{A}}(n; k_1, \ldots, k_d) = \#\big\{\{a_1, \ldots, a_d\} \in 2^{\mathcal{A}} : k_1\, a_1 + \cdots + k_d\, a_d = n\big\}?$$

3. What about an Erdős-Fuchs-type result for $k_1 = 2$ and $k_2 = 3$?

Thank you for your attention!