Computer-Assisted Proofs in Extremal Combinatorics

Workshop on Optimization and Machine Learning
Frauenhofer IIS, Waischenfeld

Christoph Spiegel (Zuse Institute Berlin)

13th of March 2023
Results are joint work with (a sunflower of)...

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ZIB / TU Berlin

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FU Berlin

Sebastian Pokutta  
ZIB / TU Berlin

Tibor Szabó  
FU Berlin

Research partially funded through Math+ projects EF1-12 and EF1-21
Computer-Assisted Proofs in Extremal Combinatorics

1. What we are interested in: A Problem of Erdős  
2. Obtaining upper bounds: Graph Blowups and Search Heuristics  
3. Obtaining lower bounds: Flag Algebras and SDPs
1. What we are interested in: *A Problem of Erdős*

**The Ramsey Multiplicity Problem**

**Theorem (Ramsey 1930)**

For any $t \in \mathbb{N}$ there exists $R_{t,t} \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R_{t,t}$ contains a monochromatic clique of size $t$.

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**Theorem (Goodman 1959 – Asymptotic Version)**

Asymptotically at least $1/4$ of all triangles are monochromatic in any 2-edge-coloring.
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**Theorem (Ramsey 1930 – Multicolor Version)**

For any \( t_1, \ldots, t_c \in \mathbb{N} \) there exists \( R_{t_1, \ldots, t_c} \in \mathbb{N} \) s.t. any \( c \)-edge-coloring of \( K_n \) with \( n \geq R_{t_1, \ldots, t_c} \in \mathbb{N} \) contains a clique of size \( t_i \) with edges colored \( i \) for some \( 1 \leq i \leq c \).

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**Beyond Goodman’s Result**

*Notation.* Let \( G_n = \{ G : E(K_n) \to [c] \} \) denote all \( c \)-edge-colorings of \( K_n \), \( G_i \) the subgraph of \( K_n \) given by color \( i \) and \( k_{t_i}(G_i) \) the fraction of \( t_i \)-cliques in \( G_i \).

**Problem (Ramsey Multiplicity)**

What is the value of \( m_{t_1, \ldots, t_c} = \lim_n \min_{G \in G_n} k_{t_1}(G_1) + \ldots + k_{t_c}(G_c) \)?

The success of the binomial random graph for \( m_{3, 3} \) lead to the following conjecture.

**Conjecture (Erdős 1962)**

\[
m_{t, t} = 2^{1 - \binom{t}{2}} \quad \text{for any } t \geq 2.
\]

False for \( t \geq 4 \) (Thomason 1989)

The exact value of even \( m_{4, 4} \) remains unknown with little progress over the last 30 years! We obtain the best current upper and lower bounds.
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2. Obtaining upper bounds: Graph Blowups and Search Heuristics

How to blow up colorings

**Notation.** Let $G^\circ_n$ denote all $c$-colorings of the **looped** $K_n$ and $k^\circ_t(G_i)$ the fraction of not nec. **injective** maps from $K_{t_i}$ to $G_i$ that are strong graph homomorphisms.

**Proposition (Bounds from any coloring)**

We have $m_{t_1,...,t_c} \leq k^\circ_t(G_1) + \ldots + k^\circ_t(G_c)$ for any $G \in G^\circ = \bigcup_n G^\circ_n$.

**Proof.** The $m$-fold blow-up $G^{\times m} \in G_{nm}$ of $G$ is obtained by replacing each vertex $v$ in $G$ with $m$ copies $v_1, \ldots, v_m$ and coloring the edge $v_iw_j$ with the color of $vw$ in $G$. By definition $m_{t_1,...,t_c} \leq \lim_{m \to \infty} k^\circ_t(G_1^{\times m}) + \ldots + k^\circ_t(G_c^{\times m}) = k^\circ_t(G_1) + \ldots + k^\circ_t(G_c)$. □

**Corollary (Relating Ramsey numbers and Ramsey multiplicity)**

By blowing up Ramsey graphs, we get $m_{t_1,...,t_c} \leq (R_{t_1,...,t_{c-1}} - 1)^{1-t_c}$. 
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Question: How can we find better candidates for $G$?
2. Obtaining upper bounds: *Graph Blowups and Search Heuristics*

### Which colorings to blow up

#### Theorem (Thomason 1989)

\[ m_{4,4} \leq 0.3050 \text{ and } m_{5,5} \leq 0.001770. \]

*Explicit by-hand construction with local search improvements.*

#### Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

\[ m_{4,4} \leq 0.03012 \text{ and } m_{5,5} \leq 0.001707. \]

*Search heuristics over Cayley graphs with specific groups.*

#### Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

\[ m_{3,4} = 689 \cdot 3^{-8} \text{ with stability results.} \]

*Search heuristics over graphs of order 27 found Schlafli graph.*

Stability proves that the search heuristic found a unique global optimum.
Which colorings to blow up

**Theorem (Franek and Rödl 1993)**

\[ m_{4,4} \leq 0.03052. \]

**Exhaustive search over specific powerset constructions.**

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Theorem (Even-Zohar and Linial '15)

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Modifying the construction of Thomason (1997).

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#### Open Problem: Do we always have

\[ m_{t_1, \ldots, t_c} = \min_{G \in G^c} k_{t_1}^c(G_1) + \ldots + k_{t_c}^c(G_c)? \]
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Razborov (2007) introduced Flag Algebras to study the limits of discrete objects.

**Definition (Flag Algebras for the empty type)**

The *flag algebra* (of the empty type) $A$ is given by considering $\mathbb{R}G$, factoring out the relations $K$ given by the *chain rule* and defining an appropriate product.

We can phrase our problem through conic optimization as

$$\max \left\{ \lambda \in \mathbb{R} : \begin{array}{c} \triangle \quad \cup \quad \lambda \emptyset \end{array} \in S = \{ f \in A : \varphi(f) \geq 0 \text{ for all } \varphi \in \text{Hom}^{+}(A, \mathbb{R}) \} \right\}$$

where $S$ is the *semantic cone* and $\text{Hom}^{+}(A, \mathbb{R}) = \{ \varphi \in \text{Hom}(A, \mathbb{R}) : \varphi|_{g \equiv 0} \}$. 

Optimizing over the semantic cone is hard. However, we can approximate it through SOS hierarchy.
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Flag Algebras and their Semantic Cones

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**Leveraging Symmetries**

The result of Goodman can be derived from the following SDP:

\[
\max_{Q \succeq 0} \min \left\{ 1 - \langle Q, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rangle, -\langle Q, \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix} \rangle, -\langle Q, \begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix} \rangle, 1 - \langle Q, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle \right\} = 1/4.
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This was obtained through computations on graphs of order \(N = 3\). Increasing \(N\) generally both improves the bound and makes the SDP harder to solve:

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**Table:** Complexity of SDP problem formulations for \(m_{4,4}\) using CSDP

How can we use combinatorial information to reduce these SDP formulations?
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**Bounds through Semidefinite Programming**

**Method 1** Reduce the number of constraints and blocks by combining constraints.

\[ \max_{Q \succeq 0} \min \left\{ 1 - \langle Q, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \rangle, - \langle Q, \begin{pmatrix} 1/6 & 1/3 \\ 1/3 & 1/6 \end{pmatrix} \rangle \right\} , \]

Uses that the Ramsey multiplicity is invariant under color permutation. Purely combinatorial proof. *Strictly stronger than considering partitions (Balogh et al. 2017).*

**Method 2** Reduce the number of variables by block diagonalization.

\[ \max_{x, y \geq 0} \min \left\{ 1 - \frac{x}{2} - \frac{y}{2}, -\frac{x}{2} + \frac{y}{6} \right\} . \]

Particularly strong when combined with Method 1. Essentially an application of Schur’s Lemma. Symmetries are easily determined combinatorially.

*Generalizes the antiinvariant split of Razborov (2010). Similar to diagonalization in SOS literature (Gatermann and Parrilo 2004). See also Bachoc et al. (2012).*
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\max_{Q \succeq 0} \min \left\{ 1 - \left\langle Q, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \right\rangle, -\left\langle Q, \begin{pmatrix} 1/6 & 1/3 \\ 1/3 & 1/6 \end{pmatrix} \right\rangle \right\},
\]


Method 2  Reduce the number of variables by block diagonalization.

\[
\max_{x,y \geq 0} \min \left\{ 1 - \frac{x}{2} - \frac{y}{6}, -\frac{x}{2} + \frac{y}{6} \right\}.
\]

Particularly strong when combined with Method 1. Essentially an application of Schur’s Lemma. Symmetries are easily determined combinatorially.

3. Obtaining lower bounds: *Flag Algebras and SDPs*

**Leveraging Symmetries**

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**Theorem (Kiem, Pokutta, S. 2022+)**

\[ m_{4,4} \geq 0.02961 \text{ and } m_{5,5} \geq 0.001557 \text{ from } N = 9. \]

---

**Theorem (Cummings et al. 2013)**

\[ m_{3,3,3} = \frac{1}{25} = \frac{1}{(R_{3,3} - 1)^2} \text{ and the only extremal constructions are based on } R_{3,3}. \]

---

**Theorem (Kiem, Pokutta, S. 2022+)**

\[ m_{3,3,3,3} = \frac{1}{256} = \frac{1}{(R_{3,3,3} - 1)^2} \text{ from } N = 6. \]

---

**Open Problem:** \( m_{3,...,3} = (R_{3,...,3} - 1)^{-2} \) for all \( c \)?
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Theorem (Kiem, Pokutta, S. 2022+)

\[ m_{3,3,3,3} \geq 1/256 - \varepsilon \text{ for some small } \varepsilon \text{ from } N = 6. \]

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Thank you for your attention!
4. Appendix

Selected related literature

4. Appendix

**Proof of Goodman’s Result**

An upper bound follows by considering the sequence of, e.g., (1) evenly-split complete bipartite graphs $K_{n/2,n/2}$ or (2) binomial random graphs $G(n,1/2)$. We saw: How to generalized the bipartite construction computationally.

A matching lower bound can symbolically be derived through

\[
\begin{align*}
\bigtriangleup + \triangle & = \frac{3}{2} \left( \left( \frac{1}{3} \bigtriangleup + \bigtriangleup \right) + \left( \frac{1}{3} \triangle + \triangle \right) - \frac{1}{3} \right) \\
& = \frac{3}{2} \left( \left( \bigtriangleup + \triangle \right) + \left( \triangle + \triangle \right) - \frac{1}{3} \right) \rightarrow \frac{3}{2} \left( \frac{2}{3}^2 + \frac{2}{3} - \frac{1}{3} \right) \\
& \geq \frac{3}{2} \left( \frac{2}{3} + \left( 1 - \frac{2}{3} \right)^2 - \frac{1}{3} \right) = 3 \left( \frac{1}{2} - \frac{1}{2} \right)^2 + \frac{1}{4} \geq \frac{1}{4}.
\end{align*}
\]

We saw: How to formalize and simplify this through Flag Algebras.
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An upper bound follows by considering the sequence of, e.g., (1) evenly-split complete bipartite graphs $K_{n/2,n/2}$ or (2) binomial random graphs $G(n,1/2)$.

We saw: How to generalized the bipartite construction computationally.

A matching lower bound can symbolically be derived through

\[
\begin{align*}
\text{upper bound} & = \frac{3}{2} \left( \left( \frac{1}{3} + \frac{1}{3} \right) + \frac{1}{3} - \frac{1}{3} \right) \\
& = \frac{3}{2} \left( \left( \frac{2}{3} + \frac{1}{3} \right) - \frac{1}{3} \right) \rightarrow \frac{3}{2} \left( 1 - \frac{1}{3} \right) \\
& \geq \frac{3}{2} \left( \left( 1 - \frac{1}{3} \right)^2 - \frac{1}{3} \right) = 3 \left( \frac{1}{3} - \frac{1}{2} \right)^2 + \frac{1}{4} \geq \frac{1}{4}.
\end{align*}
\]

We saw: How to formalize and simplify this through Flag Algebras.