Response to Reviewer 2

MATH+ Proposal: *Topological methods in Machine Learning theory*

A recent preprint by Frick, Hosseini, and Vasileuski [1] (April 2026) relates to several of the points raised by the reviewer and aligns closely with the research direction outlined in this proposal.

Q2 (Loss of information from taking the antipodal part). From the perspective of Questions 1 and 2 of the proposal (which are equivalent) there is no real loss of information. For a given class H, consider the class H^a (the *antipodal closure* of H) consisting of functions h and $-h$ for $h \in H$. Then the complex $\Delta(H^a)$ is already antipodally closed: $\Delta_{\text{ant}}(H^a) = \Delta(H^a)$. At the same time, the increase in VC dimension when passing from H to H^a is linearly bounded: $\mathrm{VC}(H^a) \leq 2\,\mathrm{VC}(H)+1$. Precisely this approach—associating to a class H the complex $\Delta(H^a)$—is taken in [1].

In what follows, classes H are taken to be antipodally closed, so that $\Delta(H) = \Delta_{\text{ant}}(H)$ is automatically antipodal.

Q1 (Why coindex?). The particular interest in coindex is inherited from the "Spherical Dimension" paper [2], where it is stated as a central question. This focus is not accidental: the dual-VC lower bounds from [3, 4] are coindex-based, and the "disambiguation of linear classifiers with margin" question from [5] directly refers to coindex (via the equivalence of Questions 1 and 2 from the proposal). The very same question is asked in [1, Question 9], phrased there as the qualitative equivalence of coindex and VC for total classes.

Coindex is not the only invariant of interest. Studying the index of these complexes is equally natural. The lower bounds in [3, 4] may be refinable in terms of index rather than coindex, potentially yielding classes whose replicability parameters substantially exceed their spherical dimension. Paralleling the coindex case, [1, Question 7] raises the question of qualitative equivalence of index and sign-rank for total classes. While there are $\mathbb{Z}_2$-spaces with coindex 1 and index arbitrarily large, no such separation is known for $\mathbb{Z}_2$-complexes arising from classes.

Regarding the other parameters suggested by the reviewer:

- **Connectivity** can be used to lower-bound the coindex, but the bound is rather conservative and does not come for free—computing connectivity may be just as challenging as computing coindex. [1] illustrates this point: its main result—a lower bound on the sign-rank of the Gap Hamming Distance partial class—is obtained by bounding the coindex via the connectivity of a particularly chosen subcomplex of Δ, a task that is challenging even for that single class. As the reviewer notes elsewhere, connectivity in such arguments is addressed via the nerve theorem.
- **Topological dimension** of Δ is just $n-1$, where n is the domain size; this is irrelevant in the typical regime $n \gg d$, where d is the VC dimension representing the "complexity" of the class.
- **Homological dimension** has not been investigated in this direction. It might be useful for upper-bounding the index, but, mirroring connectivity, such a bound is likely both too conservative and challenging to use.

Q4 (Shortcomings of "Beyond the SD approach"). The reviewer is right that considering homology spheres and nonstandard involutions was a misstep here. Such "non-standard" n-spheres still have coindex = index = n—a fact previously overlooked—and hence they do not contribute to this theory.

The intended direction is to extend the theory to invariants that more directly correspond to applications of, e.g., the local Borsuk–Ulam approach from [3]: the index mentioned above, Stiefel–Whitney height, or other refinements. Finding embeddings into Δ of n-manifolds with a coindex–index gap is a promising approach in this direction. Note, however, that such manifolds M need not admit an equivariant map $S^n \to M$, contrary to the reviewer's suggestion—i.e., they need not have coindex at least n. This serves more as an example than a primary goal of the project.

Q6 (Describe $\Delta(H)$ for certain H). The ultimate goal of the project is indeed to bring together the PAC-motivated setup with the classical tools from topological combinatorics. However, the suggestion to study $\Delta(H)$ for certain classes H is too vague to serve as a primary guiding principle, and for some classes this has already been done. In the low-dimension regime, for example, the only classes with sd $= 0$ are essentially thresholds, and all classes with sd ≥ 2 have VC ≥ 2. For a broad class of extremal classes, $\Delta(H)$ has a strong structural resemblance to the low-dimensional cube complex of H, via a deformation retraction described in [2].

The reviewer's conjecture about the realizability of all $\mathbb{Z}_2$-equivariant complexes is expected to hold for partial concept classes [5, 1]. For total classes, however—and this is also one of the points made in [1]—the situation is likely more complicated. A natural conjecture is that not only are there $\mathbb{Z}_2$-complexes non-realizable as $\Delta(H)$, but that there are topological obstructions to the construction of such complexes; for example, that the complexes $\Delta(H)$ cannot have an unbounded gap between index and coindex.

Q3 (An elaboration on the "Homological approach"). The homological approach refers to a specific construction. The size constraints of the proposal only permitted a few phrases, which ended up coming across as cryptic. Below, the construction is elaborated, with many technical details omitted to keep the exposition within reasonable bounds. No close parallels have been found in the literature, and discussions with colleagues in topological combinatorics have not produced a clear connection to standard tools.

The construction targets the setup of the main question in the form of Question 2: what is the minimal VC of a finite family $\mathcal{F}$ of sets whose antipodal interiors cover S^n? The problem can be refined as follows:

- Take S^n to be simplicial: a finite antipodal simplicial complex whose geometric realization is homeomorphic to a standard sphere S^n.
- Each $F_i \in \mathcal{F}$ is defined via its "separating hyperplane" H_i, an $(n-1)$-dimensional antipodally closed two-sided simplicial submanifold of S^n (two-sidedness can most likely be inferred from the other properties and may not need to be postulated). Then F_i and $-F_i$ correspond to different sides of H_i.
- Instead of $\mathcal{F}$, consider the family of hyperplanes H_i—an arrangement—denoted $\mathcal{H}$. Each $H \in \mathcal{H}$ splits $S^n - H$ into a positive and a negative side.
- The "cover" requirement reduces to saying that the intersection of all $H \in \mathcal{H}$ is empty.

Consider the following arrangement of four hyperplanes $H_1, \dots, H_4$ on S^2 as a guiding example.

An interesting feature of this arrangement—call it $\mathcal{H}^2$—is that its dual VC dimension is 2: $\mathrm{VC}^*(\mathcal{H}^2) = 2$. This contrasts with the naive cover $\mathcal{G}^2$ by three great circles in general position,

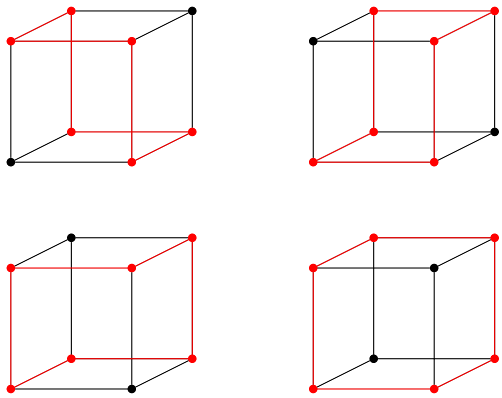

Figure 1: An alternative representation of the four hyperplanes of $\mathcal{H}^2$.

for which $\mathrm{VC}^*(\mathcal{G}^2) = 3$. This is one instance of a gap between two families of arrangements on S^n: $\mathcal{G}^n$, a naive arrangement of $(n+1)$ great circles in general position with $\mathrm{VC}^*(\mathcal{G}^n) = n+1$; and $\mathcal{H}^n$, an arrangement of 2^{n+1} hyperplanes with $\mathrm{VC}^*(\mathcal{H}^n) = \lfloor \log_2(n+1) \rfloor + 1$.

The difference in VC^* is accompanied by peculiar behavior of certain Betti numbers. For $\mathcal{G}^2 = \{G_1, G_2, G_3\}$: $\overline{G_1} = S^2 - G_1 \cong S^0$, $\overline{G_{12}} = S^2 - (G_1 \cap G_2) \cong S^1$, and $\overline{G_{123}} = S^2 - (G_1 \cap G_2 \cap G_3) = S^2 - \emptyset \cong S^2$ (the last equality because $\mathcal{G}^2$ is a cover). For $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$: $\overline{H_1} \cong S^0$, $\overline{H_{12}} \cong S^1$, $\overline{H_{123}} \cong S^1$, and $\overline{H_{1234}} \cong S^2$. The intuition—partially justified below—is that the "general position" of hyperplanes is linearly related to VC^*, and is tied to the complement of the intersection of any i hyperplanes being (homology-)equivalent to S^{i-1}.

The setup is as follows:

- A free distributive lattice $\mathbb{F} = \mathbb{F}[\mathcal{H}]$ on a finite set of generators $\mathcal{H}$. Its elements correspond to the complements of the hyperplanes in $\mathcal{H}$ and to all sets formed from them by unions and intersections.
- A function $\beta\colon \mathbb{F}[\mathcal{H}] \to (\mathbb{N} \to \mathbb{N})$ where, for $x \in \mathbb{F}$, $\beta(x)$ is finitely supported and corresponds to the sequence of (reduced) Betti numbers of the set associated with x.

These two conditions capture the basic setup. The next two are simplifying assumptions reflecting the fact that in the examples above all (complements of) intersections of hyperplanes were spheres up to homology. Let $\chi_i\colon \mathbb{N} \to \mathbb{N}$ be the sequence of (reduced) Betti numbers of S^i: $\chi_i(i) = 1$ and $\chi_i(j) = 0$ otherwise.

- For any $h \in \mathcal{H}$, $\beta(h) = \chi_0$.
- For any join g of generators from $\mathcal{H}$, $\beta(g) = \chi_i$ for some $i \in \mathbb{N}$. Then g is said to have *rank* i, denoted $\mathsf{rank}(g) = i$.

The next two conditions put natural restrictions on β. The first corresponds to exactness of the Mayer–Vietoris sequence. A finite sequence $0, n_k, n_{k-1}, \ldots, n_1, 0$ of natural numbers is *exact* if

$$n_k - n_{k-1} + n_{k-2} - \cdots + (-1)^k n_1 = 0,$$

and for all $i \in \overline{k}$

$$n_i - n_{i-1} + n_{i-2} - \cdots + (-1)^i n_1 \geq 0.$$

An infinite sequence of natural numbers is *exact* if it is finitely supported and every consecutive finite subsequence of the form $0, n_k, \ldots, n_1, 0$ is exact.

For $x, y \in \mathbb{F}$, the *MV-sequence* $\mathtt{MV}(x, y)$ is defined as

$$\ldots,\ \beta_1(x \wedge y),\ \beta_1(x) + \beta_1(y),\ \beta_1(x \vee y),\ \beta_0(x \wedge y),\ \beta_0(x) + \beta_0(y),\ \beta_0(x \vee y),\ 0.$$

The remaining conditions are:

(MV) For any $x, y \in \mathbb{F}$, the sequence $\mathtt{MV}(x, y)$ is exact.

(ND) The function β_0 is non-increasing on $\mathbb{F}$.

Recall that β_0 is the 0th reduced Betti number, i.e., the number of connected components minus 1. Condition (ND) expresses the fact that going down in $\mathbb{F}$ corresponds to removing a nowhere-dense set, which cannot kill connected components and hence cannot decrease β_0. This feature is peculiar to the total-classes setup.

The requirement that $\mathbb{F}[\mathcal{H}]$ with β actually arise from a concrete arrangement $\mathcal{H}$ is dropped from this point on, and the abstracted setting is used exclusively. The following elaborates on the connection with VC*, general position, and Betti numbers.

A subset $A \subseteq \mathcal{H}$ is *in general position* if $\beta_0(\bigwedge A) \geq 2^{|A|} - 1$. That is, general position serves as a proxy for VC*: A is in g.p. if it produces enough connected components to dually shatter A. $A \subseteq \mathcal{H}$ is *in strong general position* if, for all nonempty $B \subseteq A$, $\beta(\bigvee B) = \chi_{|B|-1}$.

The two notions coincide:

Lemma 1. *A set $A \subseteq \mathcal{H}$ is in g.p. if and only if it is in strong g.p.*

The family $\mathcal{H}^n$ above is conjectured to exhibit the smallest possible VC*. This translates into a clean conjecture in the abstracted setting:

Conjecture 1. *If $\mathbb{F}[\mathcal{H}]$ with β has rank k—i.e., $\mathsf{rank}(\bigvee \mathcal{H}) = k$—then there is $A \subseteq \mathcal{H}$ in general position of size at least $\lfloor \log_2(k+1) \rfloor + 1$.*

The validity of this approach is supported by the fact that it recovers nontrivial facts about the arrangements. The following lemma shows that $\mathcal{H}^2$ and $\mathcal{G}^2$ are essentially the only possible arrangements on S^2:

Lemma 2. *If $\mathbb{F}[\mathcal{H}]$ with β has rank 2, then either there are three elements of $\mathcal{H}$ in general position, or there are four elements $g_1, \ldots, g_4 \in \mathcal{H}$ such that, for any permutation of the g_i, $\mathsf{rank}(g_1) = 0$, $\mathsf{rank}(g_1 \vee g_2) = 1$, $\mathsf{rank}(g_1 \vee g_2 \vee g_3) = 1$, and $\mathsf{rank}(g_1 \vee g_2 \vee g_3 \vee g_4) = 2$.*

The Conjecture also holds for $n = 3$:

Lemma 3. *If $\mathbb{F}[\mathcal{H}]$ with β has rank 3, then there are three elements of $\mathcal{H}$ in general position.*

References

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