Berlin Mathematics Research Center MATH+

Flag Algebras at Scale: Opening up new Horizons

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The theory of Flag Algebras

Flag algebras are a **highly successful formalism** due to Razborov [6, 2] to study the limit behavior of combinatorial structures. For every universal first-order theory T, the fundamental object of study is a **partially ordered $\mathbb{R}$-algebra** $\mathcal{A}$ that contains all models of the theory T.

What can we define Flag Algebras for?

Theories for which flag algebras have been formulated in the past are those of **simple graphs**, vertex- or **edge-colored graphs**, **directed graphs**, oriented graphs, partially ordered sets, **k-uniform hypergraphs** and permutations of finite sets. We can also study variants of these theories, for example in the form of **triangle-free graphs** or 3-**uniform hypergraphs without independent sets of size 4**.

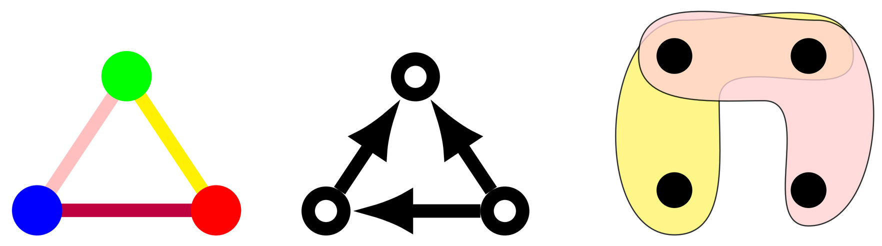


Figure 1. Some examples of combinatorial objects for which flag algebras have been explored in the past, from left to right: edge- and vertex-colored graphs, directed graphs, and 3-uniform hypergraphs.

Restating combinatorial problems through Conic Optimization

Most open problems in the area consider an element $f \in \mathcal{A}$ for a given theory T with the goal of finding a lower bound on f. Defining the *semantic cone* of $\mathcal{A}$ as

$$\mathcal{S} = \{F \in \mathcal{A} \mid \phi(F) \geq 0 \text{ for all } \phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})\}, \tag{1}$$

the optimization problem we are interested in can be stated as determining

$$\max\{\lambda \mid f - c\varnothing \in \mathcal{S} \text{ and } c \in \mathbb{R}\}. \tag{2}$$

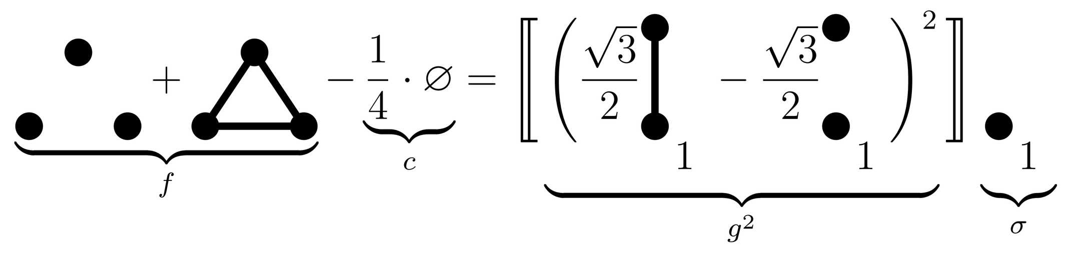
Truly optimizing over the semantic cone is computationally intractable, so **a relaxation is needed**.

Problem relaxation: sums of squares

We approximate the semantic cones $\mathcal{S}$ with **the cone given by the vectors that are sums of squares** in $\mathcal{A}$. A certificate of $f - c\varnothing \in \mathcal{S}$, i.e., $f - c \geq 0$ can be obtained from

$$f - c = \sum_{M \in \mathcal{A}} \lambda_M M + \sum_{\sigma} \sum_{i=1}^{i_\sigma} [\![g_{i,\sigma}^2]\!]_\sigma \tag{3}$$

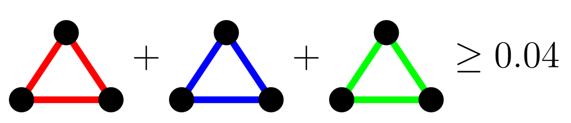
Here, the σ are fully labeled models, called **types**, inducing their own algebra $\mathcal{A}^\sigma$, in particular $\mathcal{A}^\varnothing = \mathcal{A}$ for the empty type $\varnothing$, each $g_{i,\sigma} \in \mathcal{A}^\sigma$ is an $\mathbb{R}$-linear combinations of σ-partially-labeled models, called **flags**, and $[\![\cdot]\!]_\sigma$ is the **unlabelling operator**. We can use **semi-definite programming** to determine such bounds. The asymptotic version of a famous result of Goodman [4], stating that in the limit any graph must contain at least $1/4$ of all possible cliques and independent sets of size three, can for example be obtained through the following formulation:



Flag Algebras for Edge-Colored Graphs

Previous results for three-edge-colored graphs

Cummings et al. [3] **extended the result of Goodman to three colors**, proving that



This inequality is matched by the upper bound given by blow-ups based on the 3-Ramsey graph:

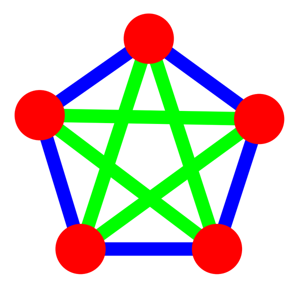
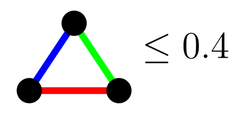


Figure 2. The unique graph on $R(3,3) - 1 = 5$ vertices not containing any triangles or their complement turned into a sequence of 3-edge-colored graphs through a blow-up construction.

Balogh et al. [1] showed that **rainbow triangles** have the following upper bound:



A matching lower bound arises from an iterated blow-up construction of the following coloring:

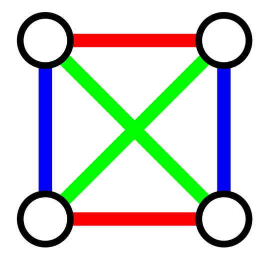
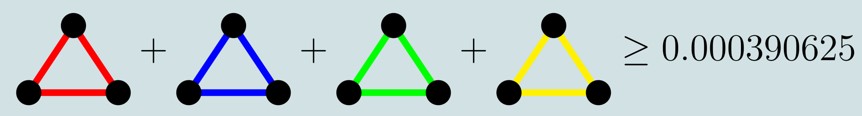


Figure 3. The base construction of the iterated blow-up sequence.

Our Contribution

We **exploit symmetries in Flag Algebra formulations in order to reduce their size** and extend the envelope of what is attainable using this technique. Using a parameter-dependent notion of homomorphisms, we can for example **reduce SDPs for c-edge-colored graphs by a factor of up to $c!$**. We also **block-diagonalize** the formulation, reducing the total number of variables.

Applying these methods, we can show for **four-edge-colored graphs** that



extending the results of Goodman and Cummings et al. This is **matched by an upper bound** based on balanced blow-up of a triangle-free Ramsey 3-edge-coloring on $R(3,3,3) - 1 = 17$ vertices. **Until now this value had been out of reach due to the size of the required SDP formulation.**

Flag Algebras in Additive Combinatorics

There has been an increased interest in **the arithmetic analogue of these questions**. Graham, Rödl, and Ruciński [5] were perhaps the first to study this, when they gave a lower bound for the number of Schur triples in 2-colorings of the integers, while Saad and Wolf [7] more recently revived interest in this area. Despite their success for graph-based theories, **Flag Algebras so far have not been explored in this setting**. We take a first step in that direction by developing the required theory in the finite-field model, that is for linear systems in $\mathbb{F}_q^n$, and applying it to some previously studied problems.

Our Contribution

We develop **two distinct notions of morphism** for colorings of $\mathbb{F}_q^n$. The first aligns with the traditional notion of homomorphisms of vector spaces and is relevant for **non-invariant additive structures such as Schur triples**. The second can be considered as an affine version and is useful when exclusively dealing with **invariant additive structures such as arithmetic progressions**.

q/n	1	2	3	4	5
2	4	8	20	92	2744
3	6	36	15 636		
4	14	7724			
5	12	72 192			

q/n	1	2	3	4	5
2	3	5	10	32	382
3	4	14	1028		
4	8	1648			
5	6	3324			

Table 1. Number of 2-colorings of $\mathbb{F}_q^n$ up to isomorphism of the first (left) and the second type (right).

Using the resulting theory, **we improve previous results and extend important ideas from the theory of graph colorings**. Most importantly, wo obtain upper and lower bounds on the number of 4-APs in colorings of $\mathbb{F}_5^n$, improving upon previous efforts of Saad and Wolf [7].

References

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