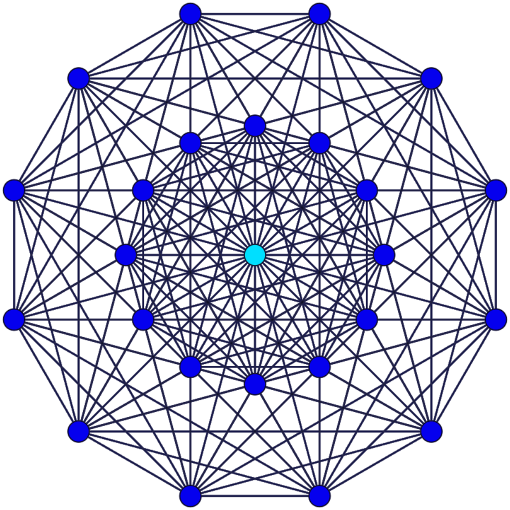
ZUSE INSTITUTE BERLIN

Fully Computer-Assisted Proofs in Extremal Combinatorics

37th AAAI Conference on Artificial Intelligence

Christoph Spiegel (Zuse Institute Berlin)

February 2023



ZUSE INSTITUTE BERLIN

Results are joint work with...

Olaf Parczyk
Freie Universität Berlin

Sebastian Pokutta
Zuse Institute Berlin

Tibor Szabó
Freie Universität Berlin

Research partially funded through Math+ project EF1-12

1. The Ramsey Multiplicity Problem
2. Search Heuristics for Constructive Bounds
3. Beyond the Ramsey multiplicity of quadrangles

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930)

For any $s, t \in \mathbb{N}$ there exists $R_{s,t} \in \mathbb{N}$ such that any graph of order at least $R_{s,t}$ contains either a clique of size s or an independent set of size t.

A well-known question

Can we determine the Ramsey numbers $R_{s,t}$ or their asymptotic behavior?

A related question

How many cliques and independent sets do we need to have?

Theorem (Goodman 1959 – asymptotic version)

Asymptotically at least $1/4$ of all triangles are either cliques or independent sets.

There has been little progress towards an answer for any $\{s, t\} \neq \{3\}$.

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930)

For any $s, t \in \mathbb{N}$ there exists $R_{s,t} \in \mathbb{N}$ such that any graph of order at least $R_{s,t}$ contains either a clique of size s or an independent set of size t.

A well-known question

Can we determine the Ramsey numbers $R_{s,t}$ or their asymptotic behavior?

A related question

How many cliques and independent sets do we need to have?

Theorem (Goodman 1959 – asymptotic version)

Asymptotically at least $1/4$ of all triangles are either cliques or independent sets.

There has been little progress towards an answer for any $\{s, t\} \neq \{3\}$.

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930)

For any $s, t \in \mathbb{N}$ there exists $R_{s,t} \in \mathbb{N}$ such that any graph of order at least $R_{s,t}$ contains either a clique of size s or an independent set of size t.

A well-known question

Can we determine the Ramsey numbers $R_{s,t}$ or their asymptotic behavior?

A related question

How many cliques and independent sets do we need to have?

Theorem (Goodman 1959 – asymptotic version)

Asymptotically at least $1/4$ of all triangles are either cliques or independent sets.

There has been little progress towards an answer for any $\{s, t\} \neq \{3\}$.

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930)

For any $s, t \in \mathbb{N}$ there exists $R_{s,t} \in \mathbb{N}$ such that any graph of order at least $R_{s,t}$ contains either a clique of size s or an independent set of size t.

A well-known question

Can we determine the Ramsey numbers $R_{s,t}$ or their asymptotic behavior?

A related question

How many cliques and independent sets do we need to have?

Theorem (Goodman 1959 – asymptotic version)

Asymptotically at least $1/4$ of all triangles are either cliques or independent sets.

There has been little progress towards an answer for any $\{s, t\} \neq \{3\}$.

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930)

For any $s, t \in \mathbb{N}$ there exists $R_{s,t} \in \mathbb{N}$ such that any graph of order at least $R_{s,t}$ contains either a clique of size s or an independent set of size t.

A well-known question

Can we determine the Ramsey numbers $R_{s,t}$ or their asymptotic behavior?

A related question

How many cliques and independent sets do we need to have?

Theorem (Goodman 1959 – asymptotic version)

Asymptotically at least $1/4$ of all triangles are either cliques or independent sets.

There has been little progress towards an answer for any $\{s, t\} \neq \{3\}$.

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n$ denote all graphs of order n and $k_s(G)$ the fraction of s-cliques in G.

Problem (Ramsey Multiplicity)

What is the value of $m_{s,t} = \lim_{n\to\infty} \min_{G\in\mathcal{G}_n} k_s(G) + k_t(\overline{G})$?

So far studied for $s = t$, though recently Behague et al. (2022+) also considered the *off-diagonal* case. For $s, t = 3$ tight upper bound given by the *binomial random graph*.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

How can we use computers to find bounds beyond human intuition?

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n$ denote all graphs of order n and $k_s(G)$ the fraction of s-cliques in G.

Problem (Ramsey Multiplicity)

What is the value of $m_{s,t} = \lim_{n\to\infty} \min_{G\in\mathcal{G}_n} k_s(G) + k_t(\overline{G})$?

So far studied for $s = t$, though recently Behague et al. (2022+) also considered the *off-diagonal* case. For $s, t = 3$ tight upper bound given by the *binomial random graph*.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

How can we use computers to find bounds beyond human intuition?

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n$ denote all graphs of order n and $k_s(G)$ the fraction of s-cliques in G.

Problem (Ramsey Multiplicity)

What is the value of $m_{s,t} = \lim_{n\to\infty} \min_{G\in\mathcal{G}_n} k_s(G) + k_t(\overline{G})$?

So far studied for $s = t$, though recently Behague et al. (2022+) also considered the *off-diagonal* case. For $s, t = 3$ tight upper bound given by the *binomial random graph*.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

How can we use computers to find bounds beyond human intuition?

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n$ denote all graphs of order n and $k_s(G)$ the fraction of s-cliques in G.

Problem (Ramsey Multiplicity)

What is the value of $m_{s,t} = \lim_{n\to\infty} \min_{G\in\mathcal{G}_n} k_s(G) + k_t(\overline{G})$?

So far studied for $s = t$, though recently Behague et al. (2022+) also considered the *off-diagonal* case. For $s, t = 3$ tight upper bound given by the *binomial random graph*.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

How can we use computers to find bounds beyond human intuition?

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n$ denote all graphs of order n and $k_s(G)$ the fraction of s-cliques in G.

Problem (Ramsey Multiplicity)

What is the value of $m_{s,t} = \lim_{n\to\infty} \min_{G\in\mathcal{G}_n} k_s(G) + k_t(\overline{G})$?

So far studied for $s = t$, though recently Behague et al. (2022+) also considered the *off-diagonal* case. For $s, t = 3$ tight upper bound given by the *binomial random graph*.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

How can we use computers to find bounds beyond human intuition?

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n$ denote all graphs of order n and $k_s(G)$ the fraction of s-cliques in G.

Problem (Ramsey Multiplicity)

What is the value of $m_{s,t} = \lim_{n\to\infty} \min_{G\in\mathcal{G}_n} k_s(G) + k_t(\overline{G})$?

So far studied for $s = t$, though recently Behague et al. (2022+) also considered the *off-diagonal* case. For $s, t = 3$ tight upper bound given by the *binomial random graph*.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

How can we use computers to find bounds beyond human intuition?

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n$ denote all graphs of order n and $k_s(G)$ the fraction of s-cliques in G.

Problem (Ramsey Multiplicity)

What is the value of $m_{s,t} = \lim_{n\to\infty} \min_{G\in\mathcal{G}_n} k_s(G) + k_t(\overline{G})$?

So far studied for $s = t$, though recently Behague et al. (2022+) also considered the *off-diagonal* case. For $s, t = 3$ tight upper bound given by the *binomial random graph*.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

How can we use computers to find bounds beyond human intuition?

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n$ denote all graphs of order n and $k_s(G)$ the fraction of s-cliques in G.

Problem (Ramsey Multiplicity)

What is the value of $m_{s,t} = \lim_{n\to\infty} \min_{G\in\mathcal{G}_n} k_s(G) + k_t(\overline{G})$?

So far studied for $s = t$, though recently Behague et al. (2022+) also considered the *off-diagonal* case. For $s, t = 3$ tight upper bound given by the *binomial random graph*.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

How can we use computers to find **constructive** bounds beyond human intuition?

1. The Ramsey Multiplicity Problem

2. Search Heuristics for Constructive Bounds

3. Beyond the Ramsey multiplicity of quadrangles

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Corollary (Relating Ramsey numbers and Ramsey multiplicity)

By blowing up Ramsey graphs, we get $m_{s,t} \leq (R_{s,t} - 1)^{1-t}$.

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Corollary (Relating Ramsey numbers and Ramsey multiplicity)

By blowing up Ramsey graphs, we get $m_{s,t} \leq (R_{s,t} - 1)^{1-t}$.

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Corollary (Relating Ramsey numbers and Ramsey multiplicity)

By blowing up Ramsey graphs, we get $m_{s,t} \leq (R_{s,t} - 1)^{1-t}$.

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Corollary (Relating Ramsey numbers and Ramsey multiplicity)

By blowing up Ramsey graphs, we get $m_{s,t} \leq (R_{s,t} - 1)^{1-t}$.

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Corollary (Relating Ramsey numbers and Ramsey multiplicity)

By blowing up Ramsey graphs, we get $m_{s,t} \leq (R_{s,t} - 1)^{1-t}$.

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Question: How can we find better candidates for G?

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Theorem (Thomason 1989)

$m_{4,4} \leq 0.3050$ and $m_{5,5} \leq 0.001770$.

Explicit by-hand construction with local search improvements.

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \dots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Theorem (Franek and Rödl 1993)

$m_{4,4} \leq 0.03052$ (not an improvement)

Exhaustive search over specific powerset constructions.

ZUSE INSTITUTE BERLIN

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Theorem (Thomason 1997)

$m_{4,4} \leq 0.03031$ and $m_{5,5} \leq 0.001720$.

Exhaustive search over small XOR graph products.

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Theorem (Even-Zohar and Linial '15)

$m_{4,4} \leq 0.03028$.

Iterating the construction of Thomason (1997).

How to blow up graphs

Notation. Let $\mathcal{G}_n^\circ$ denote all possibly looped graphs of order n and $k_s^\circ(G)$ the fraction of **not nec. injective** maps from K_s to G that are strong graph homomorphisms.

Proposition (Bounds from any graph)

We have $m_{s,t} \leq k_s^\circ(G) + k_t^\circ(\overline{G})$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and connecting v_i with w_j in $G^{\times m}$ if v is adjacent to w in G. By definition $m_{s,t} \leq \lim_{m\to\infty} k_s(G^{\times m}) + k_t(\overline{G^{\times m}}) = k_s^\circ(G) + k_t^\circ(\overline{G})$.* □

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{4,4} \leq 0.03012$ and $m_{5,5} \leq 0.001707$.

Search heuristics over Cayley graph search space ...

Constructing graphs through search heuristics

We phrased the problem through binary representations and applied Metaheuristics, i.e., Simulated Annealing (Kirkpatrick et al. 1983) and Tabu search (Glover 1986):

For any binary vector $\mathbf{x} = (x_1, \ldots, x_{\binom{n}{2}+n}) \in \{0,1\}^{\binom{n}{2}+n}$ let the graph $G_{\mathbf{x}} \in \mathcal{G}_n^\circ$ be given by connecting any two vertices $1 \leq i \leq j \leq n$ if $x_{\binom{j}{2}+i} = 1$. We want to determine

$$\min_{\mathbf{x}} k_s^\circ(G_{\mathbf{x}}) + k_t^\circ(\overline{G_{\mathbf{x}}}). \tag{1}$$

Disadvantages. Problem structure is ignored. No guarantees of optimality.

Advantages. Applicable to many problems. Optimality through matching bounds.

SA and TS have previously proven successful in finding Ramsey numbers. Recently learning-based approaches have also been suggested (Wagner 2021; Bello et al. 2016).

Constructing graphs through search heuristics

We phrased the problem through binary representations and applied Metaheuristics, i.e., Simulated Annealing (Kirkpatrick et al. 1983) and Tabu search (Glover 1986):

For any binary vector $\mathbf{x} = (x_1, \ldots, x_{\binom{n}{2}+n}) \in \{0,1\}^{\binom{n}{2}+n}$ let the graph $G_{\mathbf{x}} \in \mathcal{G}_n^\circ$ be given by connecting any two vertices $1 \leq i \leq j \leq n$ if $x_{\binom{j}{2}+i} = 1$. We want to determine

$$\min_{\mathbf{x}} k_s^\circ(G_{\mathbf{x}}) + k_t^\circ(\overline{G_{\mathbf{x}}}). \tag{1}$$

Disadvantages. Problem structure is ignored. No guarantees of optimality.

Advantages. Applicable to many problems. Optimality through matching bounds.

SA and TS have previously proven successful in finding Ramsey numbers. Recently learning-based approaches have also been suggested (Wagner 2021; Bello et al. 2016).

Constructing graphs through search heuristics

We phrased the problem through binary representations and applied Metaheuristics, i.e., Simulated Annealing (Kirkpatrick et al. 1983) and Tabu search (Glover 1986):

For any binary vector $\mathbf{x} = (x_1, \ldots, x_{\binom{n}{2}+n}) \in \{0,1\}^{\binom{n}{2}+n}$ let the graph $G_{\mathbf{x}} \in \mathcal{G}_n^{\circ}$ be given by connecting any two vertices $1 \leq i \leq j \leq n$ if $x_{\binom{j}{2}+i} = 1$. We want to determine

$$\min_{\mathbf{x}} k_s^{\circ}(G_{\mathbf{x}}) + k_t^{\circ}(\overline{G_{\mathbf{x}}}). \tag{1}$$

Disadvantages. Problem structure is ignored. No guarantees of optimality.

Advantages. Applicable to many problems. Optimality through matching bounds.

SA and TS have previously proven successful in finding Ramsey numbers. Recently learning-based approaches have also been suggested (Wagner 2021; Bello et al. 2016).

Constructing graphs through search heuristics

We phrased the problem through binary representations and applied Metaheuristics, i.e., Simulated Annealing (Kirkpatrick et al. 1983) and Tabu search (Glover 1986):

For any binary vector $\mathbf{x} = (x_1, \ldots, x_{\binom{n}{2}+n}) \in \{0,1\}^{\binom{n}{2}+n}$ let the graph $G_{\mathbf{x}} \in \mathcal{G}_n^{\circ}$ be given by connecting any two vertices $1 \leq i \leq j \leq n$ if $x_{\binom{j}{2}+i} = 1$. We want to determine

$$\min_{\mathbf{x}} k_s^{\circ}(G_{\mathbf{x}}) + k_t^{\circ}(\overline{G_{\mathbf{x}}}). \tag{1}$$

Disadvantages. Problem structure is ignored. No guarantees of optimality.

Advantages. Applicable to many problems. Optimality through matching bounds.

SA and TS have previously proven successful in finding Ramsey numbers. Recently learning-based approaches have also been suggested (Wagner 2021; Bello et al. 2016).

Constructing graphs through search heuristics

We phrased the problem through binary representations and applied Metaheuristics, i.e., Simulated Annealing (Kirkpatrick et al. 1983) and Tabu search (Glover 1986):

For any binary vector $\mathbf{x} = (x_1, \ldots, x_{\binom{n}{2}+n}) \in \{0,1\}^{\binom{n}{2}+n}$ let the graph $G_{\mathbf{x}} \in \mathcal{G}_n^\circ$ be given by connecting any two vertices $1 \leq i \leq j \leq n$ if $x_{\binom{j}{2}+i} = 1$. We want to determine

$$\min_{\mathbf{x}} k_s^\circ(G_{\mathbf{x}}) + k_t^\circ(\overline{G_{\mathbf{x}}}). \tag{1}$$

Disadvantages. Problem structure is ignored. No guarantees of optimality.

Advantages. Applicable to many problems. Optimality through matching bounds.

SA and TS have previously proven successful in finding Ramsey numbers. Recently learning-based approaches have also been suggested (Wagner 2021; Bello et al. 2016).

Constructing Cayley graphs through search heuristics

Unfortunately does not scale beyond $n \approx 40$, barely disproving Erdős' original conjecture for $m_{4,4}$. **Can we bias the search space using combinatorial insights?**

Turns out all previous constructions are actually graphs with very specific symmetries:

Definition (Cayley graphs)

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Let $\mathbf{x}$ now represent the generating set S. Since $|G|/2 < |S| < |G|$ the number of binary variables is linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for $m_{4,4}$ and $m_{5,5}$.

Constructing Cayley graphs through search heuristics

Unfortunately does not scale beyond $n \approx 40$, barely disproving Erdős' original conjecture for $m_{4,4}$. **Can we bias the search space using combinatorial insights?**

Turns out all previous constructions are actually graphs with very specific symmetries:

Definition (Cayley graphs)

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Let $\mathbf{x}$ now represent the generating set S. Since $|G|/2 < |S| < |G|$ the number of binary variables is linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for $m_{4,4}$ and $m_{5,5}$.

Constructing Cayley graphs through search heuristics

Unfortunately does not scale beyond $n \approx 40$, barely disproving Erdős' original conjecture for $m_{4,4}$. **Can we bias the search space using combinatorial insights?**

Turns out all previous constructions are actually graphs with very specific symmetries:

Definition (Cayley graphs)

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Let $\mathbf{x}$ now represent the generating set S. Since $|G|/2 < |S| < |G|$ the number of binary variables is linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for $m_{4,4}$ and $m_{5,5}$.

Constructing Cayley graphs through search heuristics

Unfortunately does not scale beyond $n \approx 40$, barely disproving Erdős' original conjecture for $m_{4,4}$. **Can we bias the search space using combinatorial insights?**

Turns out all previous constructions are actually graphs with very specific symmetries:

Definition (Cayley graphs)

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Let $\mathbf{x}$ now represent the generating set S. Since $|G|/2 < |S| < |G|$ the number of binary variables is linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for $m_{4,4}$ and $m_{5,5}$.

Constructing Cayley graphs through search heuristics

Unfortunately does not scale beyond $n \approx 40$, barely disproving Erdős' original conjecture for $m_{4,4}$. **Can we bias the search space using combinatorial insights?**

Turns out all previous constructions are actually graphs with very specific symmetries:

Definition (Cayley graphs)

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Let $\mathbf{x}$ now represent the generating set S. Since $|G|/2 < |S| < |G|$ the number of binary variables is linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for $m_{4,4}$ and $m_{5,5}$.

Constructing Cayley graphs through search heuristics

Unfortunately does not scale beyond $n \approx 40$, barely disproving Erdős' original conjecture for $m_{4,4}$. **Can we bias the search space using combinatorial insights?**

Turns out all previous constructions are actually graphs with very specific symmetries:

Definition (Cayley graphs)

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Let $\mathbf{x}$ now represent the generating set S. Since $|G|/2 < |S| < |G|$ the number of binary variables is linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for $m_{4,4}$ and $m_{5,5}$.

Off-diagonal Ramsey Multiplicity

Question. Determining $m_{3,3}$ is easy, but even $m_{4,4}$ has been unresolved for over 60 years. Can we say more when studying the off-diagonal variant where $s \neq t$?

A famous result of Reiher from 2016 implies that $m_{2,t} = 1/(t-1)$.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ and any large enough graph G admits a strong homomorphism into the Schläfli graph after changing at most $O(k_3(\overline{G}) + k_4(G) - m_{3,4})\, v(G)^2$ edges.

The fact that we can show stability proves that the search heuristic found a unique global optimum over all graphs of order 27!

Off-diagonal Ramsey Multiplicity

Question. Determining $m_{3,3}$ is easy, but even $m_{4,4}$ has been unresolved for over 60 years. Can we say more when studying the off-diagonal variant where $s \neq t$?

A famous result of Reiher from 2016 implies that $m_{2,t} = 1/(t-1)$.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ and any large enough graph G admits a strong homomorphism into the Schläfli graph after changing at most $O(k_3(\overline{G}) + k_4(G) - m_{3,4})\, v(G)^2$ edges.

The fact that we can show stability proves that the search heuristic found a unique global optimum over all graphs of order 27!

Off-diagonal Ramsey Multiplicity

Question. Determining $m_{3,3}$ is easy, but even $m_{4,4}$ has been unresolved for over 60 years. Can we say more when studying the off-diagonal variant where $s \neq t$?

A famous result of Reiher from 2016 implies that $m_{2,t} = 1/(t-1)$.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ and any large enough graph G admits a strong homomorphism into the Schläfli graph after changing at most $O(k_3(\overline{G}) + k_4(G) - m_{3,4})\, v(G)^2$ edges.

The fact that we can show stability proves that the search heuristic found a unique global optimum over all graphs of order 27!

Off-diagonal Ramsey Multiplicity

Question. Determining $m_{3,3}$ is easy, but even $m_{4,4}$ has been unresolved for over 60 years. Can we say more when studying the off-diagonal variant where $s \neq t$?

A famous result of Reiher from 2016 implies that $m_{2,t} = 1/(t-1)$.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ and any large enough graph G admits a strong homomorphism into the Schläfli graph after changing at most $O(k_3(\overline{G}) + k_4(G) - m_{3,4})\, v(G)^2$ edges.

The fact that we can show stability proves that the search heuristic found a unique global optimum over all graphs of order 27!

Remarks and Open Problems

- One can also study other variants of the Ramsey multiplicity problem, for example for hypergraphs, additive structures, structures in finite geometry ...

- For many other problems the (current best) upper bounds come from blow-up-esque constructions: the capset problem, the Sunflower conjecture, Turán's $(3,4)$-conjecture, the Shannon Capacity of odd cycles ...

- One can more fundamentally ask when such constructions are optimal, e.g., do we always have $m_{s,t} = \min_{G \in \mathcal{G}^\circ} k_s^\circ(G) + k_t^\circ(\overline{G})$ for the Ramsey multiplicity problem?

Remarks and Open Problems

- One can also study other variants of the Ramsey multiplicity problem, for example for hypergraphs, additive structures, structures in finite geometry ...
- For many other problems the (current best) upper bounds come from blow-up-esque constructions: the capset problem, the Sunflower conjecture, Turán's $(3, 4)$-conjecture, the Shannon Capacity of odd cycles ...
- One can more fundamentally ask when such constructions are optimal, e.g., do we always have $m_{s,t} = \min_{G \in \mathcal{G}^\circ} k_s^\circ(G) + k_t^\circ(\overline{G})$ for the Ramsey multiplicity problem?

Remarks and Open Problems

- One can also study other variants of the Ramsey multiplicity problem, for example for hypergraphs, additive structures, structures in finite geometry ...

- For many other problems the (current best) upper bounds come from blow-up-esque constructions: the capset problem, the Sunflower conjecture, Turán's $(3,4)$-conjecture, the Shannon Capacity of odd cycles ...

- One can more fundamentally ask when such constructions are optimal, e.g., do we always have $m_{s,t} = \min_{G \in \mathcal{G}^\circ} k_s^\circ(G) + k_t^\circ(\overline{G})$ for the Ramsey multiplicity problem?

Thank you for your attention!