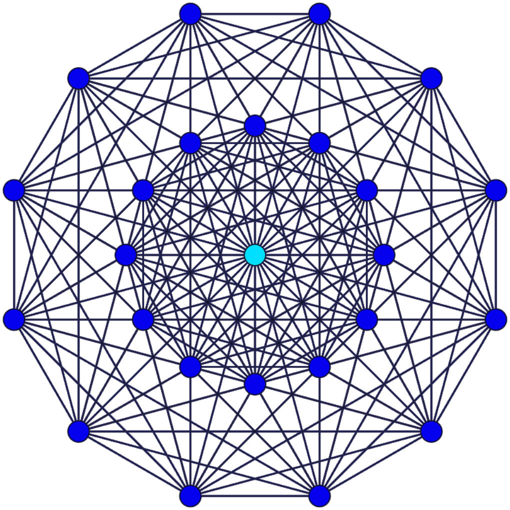
Fully Computer-Assisted Proofs in Extremal Combinatorics

25th Combinatorial Optimization Workshop
CNRS Centre Paul Langevin, Aussois

Christoph Spiegel (Zuse Institute Berlin)

9th to 13th of January 2023



Results are joint work with (a sunflower of)...

Aldo Kiem
ZIB / TU Berlin

Olaf Parczyk
FU Berlin

Sebastian Pokutta
ZIB / TU Berlin

Tibor Szabó
FU Berlin

Research partially funded through Math+ projects EF1-12 and EF1-21

Computer-Assisted Proofs in Extremal Combinatorics

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930)

For any $t \in \mathbb{N}$ there exists $R_{t,t} \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R_{t,t}$ contains a monochromatic clique of size t.

A well-known question

Can we determine $R_{t_1,\dots,t_c}$?

A related question

How many cliques are required?

Theorem (Goodman 1959 – Asymptotic Version)

Asymptotically at least $1/4$ of all triangles are monochromatic in any 2-edge-coloring.

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930 – Multicolor Version)

For any $t_1, \dots, t_c \in \mathbb{N}$ there exists $R_{t_1,\dots,t_c} \in \mathbb{N}$ s.t. any c-edge-coloring of K_n with $n \geq R_{t_1,\dots,t_c} \in \mathbb{N}$ contains an clique of size t_i with edges colored i for some $1 \leq i \leq c$.

A well-known question

Can we determine $R_{t_1,\dots,t_c}$?

A related question

How many cliques are required?

Theorem (Goodman 1959 – Asymptotic Version)

Asymptotically at least $1/4$ of all triangles are monochromatic in any 2-edge-coloring.

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930 – Multicolor Version)

For any $t_1, \dots, t_c \in \mathbb{N}$ there exists $R_{t_1,\dots,t_c} \in \mathbb{N}$ s.t. any c-edge-coloring of K_n with $n \geq R_{t_1,\dots,t_c} \in \mathbb{N}$ contains an clique of size t_i with edges colored i for some $1 \leq i \leq c$.

A well-known question

Can we determine $R_{t_1,\dots,t_c}$?

A related question

How many cliques are required?

Theorem (Goodman 1959 – Asymptotic Version)

Asymptotically at least $1/4$ of all triangles are monochromatic in any 2-edge-coloring.

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930 – Multicolor Version)

For any $t_1, \ldots, t_c \in \mathbb{N}$ there exists $R_{t_1,\ldots,t_c} \in \mathbb{N}$ s.t. any c-edge-coloring of K_n with $n \geq R_{t_1,\ldots,t_c} \in \mathbb{N}$ contains an clique of size t_i with edges colored i for some $1 \leq i \leq c$.

A well-known question

Can we determine $R_{t_1,\ldots,t_c}$?

A related question

How many cliques are required?

Theorem (Goodman 1959 – Asymptotic Version)

Asymptotically at least $1/4$ of all triangles are monochromatic in any 2-edge-coloring.

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n = \{G : E(K_n) \to [c]\}$ denote all c-edge-colorings of K_n, G_i the subgraph of K_n given by color i and $k_{t_i}(G_i)$ the fraction of t_i-cliques in G_i.

Problem (Ramsey Multiplicity)

What is the value of $m_{t_1,\ldots,t_c} = \lim_n \min_{G \in \mathcal{G}_n} k_{t_1}(G_1) + \ldots + k_{t_c}(G_c)$?

The success of the binomial random graph for $m_{3,3}$ lead to the following conjecture.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

The exact value of even $m_{4,4}$ remains unknown with little progress over the last 30 years! We obtain the best current upper and lower bounds.

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n = \{G : E(K_n) \to [c]\}$ denote all c-edge-colorings of K_n, G_i the subgraph of K_n given by color i and $k_{t_i}(G_i)$ the fraction of t_i-cliques in G_i.

Problem (Ramsey Multiplicity)

What is the value of $m_{t_1,\ldots,t_c} = \lim_n \min_{G\in\mathcal{G}_n} k_{t_1}(G_1) + \ldots + k_{t_c}(G_c)$?

The success of the binomial random graph for $m_{3,3}$ lead to the following conjecture.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

The exact value of even $m_{4,4}$ remains unknown with little progress over the last 30 years! We obtain the best current upper and lower bounds.

Beyond Goodman's Result

Notation. Let $\mathcal{G}_n = \{G : E(K_n) \to [c]\}$ denote all c-edge-colorings of K_n, G_i the subgraph of K_n given by color i and $k_{t_i}(G_i)$ the fraction of t_i-cliques in G_i.

Problem (Ramsey Multiplicity)

What is the value of $m_{t_1,\ldots,t_c} = \lim_n \min_{G \in \mathcal{G}_n} k_{t_1}(G_1) + \ldots + k_{t_c}(G_c)$?

The success of the binomial random graph for $m_{3,3}$ lead to the following conjecture.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

The exact value of even $m_{4,4}$ remains unknown with little progress over the last 30 years! We obtain the best current upper and lower bounds.

Computer-Assisted Proofs in Extremal Combinatorics

How to blow up colorings

Notation. Let $\mathcal{G}_n^\circ$ denote all c-colorings of the **looped** K_n and $k_{t_i}^\circ(G_i)$ the fraction of **not nec. injective** maps from K_{t_i} to G_i that are strong graph homomorphisms.

Proposition (Bounds from any coloring)

We have $m_{t_1,\ldots,t_c} \leq k_{t_1}^\circ(G_1) + \ldots + k_{t_c}^\circ(G_c)$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and coloring the edge $v_i w_j$ with the color of vw in G. By definition* $m_{t_1,\ldots,t_c} \leq \lim_{m\to\infty} k_{t_1}(G_1^{\times m}) + \ldots + k_{t_c}(G_c^{\times m}) = k_{t_1}^\circ(G_1) + \ldots + k_{t_c}^\circ(G_c)$. □

Corollary (Relating Ramsey numbers and Ramsey multiplicity)

By blowing up Ramsey graphs, we get $m_{t_1,\ldots,t_c} \leq (R_{t_1,\ldots,t_{c-1}} - 1)^{1-t_c}$.

How to blow up colorings

Notation. Let $\mathcal{G}_n^\circ$ denote all c-colorings of the **looped** K_n and $k_{t_i}^\circ(G_i)$ the fraction of **not nec. injective** maps from K_{t_i} to G_i that are strong graph homomorphisms.

Proposition (Bounds from any coloring)

We have $m_{t_1,\ldots,t_c} \leq k_{t_1}^\circ(G_1) + \ldots + k_{t_c}^\circ(G_c)$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and coloring the edge $v_i w_j$ with the color of vw in G. By definition* $m_{t_1,\ldots,t_c} \leq \lim_{m\to\infty} k_{t_1}(G_1^{\times m}) + \ldots + k_{t_c}(G_c^{\times m}) = k_{t_1}^\circ(G_1) + \ldots + k_{t_c}^\circ(G_c)$. □

Corollary (Relating Ramsey numbers and Ramsey multiplicity)

By blowing up Ramsey graphs, we get $m_{t_1,\ldots,t_c} \leq (R_{t_1,\ldots,t_{c-1}} - 1)^{1-t_c}$.

How to blow up colorings

Notation. Let $\mathcal{G}_n^\circ$ denote all c-colorings of the **looped** K_n and $k_{t_i}^\circ(G_i)$ the fraction of **not nec. injective** maps from K_{t_i} to G_i that are strong graph homomorphisms.

Proposition (Bounds from any coloring)

We have $m_{t_1,\ldots,t_c} \leq k_{t_1}^\circ(G_1) + \ldots + k_{t_c}^\circ(G_c)$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and coloring the edge $v_i w_j$ with the color of vw in G. By definition* $m_{t_1,\ldots,t_c} \leq \lim_{m\to\infty} k_{t_1}(G_1^{\times m}) + \ldots + k_{t_c}(G_c^{\times m}) = k_{t_1}^\circ(G_1) + \ldots + k_{t_c}^\circ(G_c)$. $\square$

Corollary (Relating Ramsey numbers and Ramsey multiplicity)

By blowing up Ramsey graphs, we get $m_{t_1,\ldots,t_c} \leq (R_{t_1,\ldots,t_{c-1}} - 1)^{1-t_c}$.

How to blow up colorings

Notation. Let $\mathcal{G}_n^\circ$ denote all c-colorings of the **looped** K_n and $k_{t_i}^\circ(G_i)$ the fraction of **not nec. injective** maps from K_{t_i} to G_i that are strong graph homomorphisms.

Proposition (Bounds from any coloring)

We have $m_{t_1,\dots,t_c} \leq k_{t_1}^\circ(G_1) + \ldots + k_{t_c}^\circ(G_c)$ for **any** $G \in \mathcal{G}^\circ = \bigcup_n \mathcal{G}_n^\circ$.

Proof. *The m-fold blow-up $G^{\times m} \in \mathcal{G}_{nm}$ of G is obtained by replacing each vertex v in G with m copies $v_1, \ldots, v_m$ and coloring the edge $v_i w_j$ with the color of vw in G. By definition $m_{t_1,\dots,t_c} \leq \lim_{m\to\infty} k_{t_1}(G_1^{\times m}) + \ldots + k_{t_c}(G_c^{\times m}) = k_{t_1}^\circ(G_1) + \ldots + k_{t_c}^\circ(G_c)$.* $\square$

Question: How can we find better candidates for G?

Which colorings to blow up

Theorem (Thomason 1989)

$m_{4,4} \leq 0.3050$ and $m_{5,5} \leq 0.001770$.

Explicit by-hand construction with local search improvements.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{4,4} \leq 0.03012$ and $m_{5,5} \leq 0.001707$.

Search heuristics over Cayley graphs with specific groups.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ with stability results.

Search heuristics over graphs of order 27 found Schläfli graph.

Stability proves that the search heuristic found a unique global optimum.

Which colorings to blow up

Theorem (Franek and Rödl 1993)

$m_{4,4} \leq 0.03052.$

Exhaustive search over specific powerset constructions.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{4,4} \leq 0.03012$ and $m_{5,5} \leq 0.001707.$

Search heuristics over Cayley graphs with specific groups.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ with stability results.

Search heuristics over graphs of order 27 found Schläfli graph.

Stability proves that the search heuristic found a unique global optimum.

Which colorings to blow up

Theorem (Thomason 1997)

$m_{4,4} \leq 0.03031$ and $m_{5,5} \leq 0.001720$.

Exhaustive search over XOR graph products.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{4,4} \leq 0.03012$ and $m_{5,5} \leq 0.001707$.

Search heuristics over Cayley graphs with specific groups.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ with stability results.

Search heuristics over graphs of order 27 found Schläfli graph.

Stability proves that the search heuristic found a unique global optimum.

Which colorings to blow up

Theorem (Even-Zohar and Linial '15)

$m_{4,4} \leq 0.03028$.

Iterating the construction of Thomason (1997).

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{4,4} \leq 0.03012$ and $m_{5,5} \leq 0.001707$.

Search heuristics over Cayley graphs with specific groups.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ with stability results.

Search heuristics over graphs of order 27 found Schläfli graph.

Stability proves that the search heuristic found a unique global optimum.

Which colorings to blow up

Theorem (Even-Zohar and Linial '15)

$m_{4,4} \leq 0.03028$.

Iterating the construction of Thomason (1997).

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{4,4} \leq 0.03012$ and $m_{5,5} \leq 0.001707$.

Search heuristics over Cayley graphs with specific groups.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ with stability results.

Search heuristics over graphs of order 27 found Schläfli graph.

Stability proves that the search heuristic found a unique global optimum.

Which colorings to blow up

Theorem (Even-Zohar and Linial '15)

$m_{4,4} \leq 0.03028$.

Iterating the construction of Thomason (1997).

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{4,4} \leq 0.03012$ and $m_{5,5} \leq 0.001707$.

Search heuristics over Cayley graphs with specific groups.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ with stability results.

Search heuristics over graphs of order 27 found Schläfli graph.

Stability proves that the search heuristic found a unique global optimum.

Which colorings to blow up

Theorem (Even-Zohar and Linial '15)

$m_{4,4} \leq 0.03028$.

Iterating the construction of Thomason (1997).

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{4,4} \leq 0.03012$ and $m_{5,5} \leq 0.001707$.

Search heuristics over Cayley graphs with specific groups.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$m_{3,4} = 689 \cdot 3^{-8}$ with stability results.

Search heuristics over graphs of order 27 found Schläfli graph.

Open Problem: Do we always have $m_{t_1,\dots,t_c} = \min_{G \in \mathcal{G}^\circ} k^\circ_{t_1}(G_1) + \ldots + k^\circ_{t_c}(G_c)$?

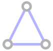
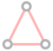
Flag Algebras and their Semantic Cones

Razborov (2007) introduced Flag Algebras to study the limits of discrete objects.

Definition (Flag Algebras for the empty type)

The *flag algebra* (of the empty type) $\mathcal{A}$ is given by considering $\mathbb{R}\mathcal{G}$, factoring out the relations $\mathcal{K}$ given by the *chain rule* and defining an appropriate product.

We can phrase our problem through conic optimization as

$\max \Big\{ \lambda \in \mathbb{R} :$  $+$  $- \lambda \varnothing \in \mathcal{S} = \{ f \in \mathcal{A} : \varphi(f) \geq 0 \text{ for all } \varphi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R}) \} \Big\}$

where $\mathcal{S}$ is the *semantic cone* and $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R}) = \{ \varphi \in \mathrm{Hom}(\mathcal{A}, \mathbb{R}) : \varphi|_{\mathcal{G}} \equiv 0 \}$.

Optimizing over the semantic cone is hard.
However, we can approximate it through SOS hierarchy.

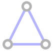
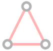
Flag Algebras and their Semantic Cones

Razborov (2007) introduced Flag Algebras to study the limits of discrete objects.

Definition (Flag Algebras for the empty type)

The *flag algebra* (of the empty type) $\mathcal{A}$ is given by considering $\mathbb{R}\mathcal{G}$, factoring out the relations $\mathcal{K}$ given by the *chain rule* and defining an appropriate product.

We can phrase our problem through conic optimization as

$\max \Big\{ \lambda \in \mathbb{R} :$  $+$  $- \lambda \varnothing \in \mathcal{S} = \{ f \in \mathcal{A} : \varphi(f) \geq 0$ for all $\varphi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})\} \Big\}$

where $\mathcal{S}$ is the *semantic cone* and $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R}) = \{\varphi \in \mathrm{Hom}(\mathcal{A}, \mathbb{R}) : \varphi|_{\mathcal{G}} \equiv 0\}$.

Optimizing over the semantic cone is hard.
However, we can approximate it through SOS hierarchy.

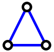
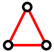
Flag Algebras and their Semantic Cones

Razborov (2007) introduced Flag Algebras to study the limits of discrete objects.

Definition (Flag Algebras for the empty type)

The *flag algebra* (of the empty type) $\mathcal{A}$ is given by considering $\mathbb{R}\mathcal{G}$, factoring out the relations $\mathcal{K}$ given by the *chain rule* and defining an appropriate product.

We can phrase our problem through conic optimization as

max $\Big\{\lambda \in \mathbb{R}$:  $+$  $- \lambda\varnothing \in \mathcal{S} = \{f \in \mathcal{A} : \varphi(f) \geq 0$ for all $\varphi \in \mathrm{Hom}^{+}(\mathcal{A}, \mathbb{R})\}\Big\}$

where $\mathcal{S}$ is the *semantic cone* and $\mathrm{Hom}^{+}(\mathcal{A}, \mathbb{R}) = \{\varphi \in \mathrm{Hom}(\mathcal{A}, \mathbb{R}) : \varphi|_{\mathcal{G}} \equiv 0\}$.

Optimizing over the semantic cone is hard.
However, we can approximate it through SOS hierarchy.

Leveraging Symmetries

The result of Goodman can be derived from the following SDP:

$$\max_{Q\succeq 0}\min\left\{1-\left\langle Q,\begin{pmatrix}1&0\\0&0\end{pmatrix}\right\rangle,-\left\langle Q,\begin{pmatrix}1/3&1/3\\1/3&0\end{pmatrix}\right\rangle,-\left\langle Q,\begin{pmatrix}0&1/3\\1/3&1/3\end{pmatrix}\right\rangle,1-\left\langle Q,\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\rangle\right\}=1/4.$$

This was obtained through computations on graphs of order $N = 3$. Increasing N generally both improves the bound and makes the SDP harder to solve:

N	*value*	*time*	*memory*
6	0.02875	0.2s ±0.0	81.2MB ±24.7
7	0.02918	4.9s ±0.1	126.9MB ±26.3
8	0.02942	1.8h ±0.1	1.8GB ±0.0

Table: Complexity of SDP problem formulations for $m_{4,4}$ using CSDP

How can we use combinatorial information to reduce these SDP formulations?

Leveraging Symmetries

The result of Goodman can be derived from the following SDP:

$$\max_{Q \succeq 0} \min \left\{ 1 - \left\langle Q, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle, - \left\langle Q, \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix} \right\rangle, - \left\langle Q, \begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix} \right\rangle, 1 - \left\langle Q, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \right\} = 1/4.$$

This was obtained through computations on graphs of order $N = 3$. Increasing N generally both improves the bound and makes the SDP harder to solve:

N	*value*	*time*	*memory*
6	0.02875	0.2s ± 0.0	81.2MB ± 24.7
7	0.02918	4.9s ± 0.1	126.9MB ± 26.3
8	0.02942	1.8h ± 0.1	1.8GB ± 0.0

Table: Complexity of SDP problem formulations for $m_{4,4}$ using `CSDP`

How can we use combinatorial information to reduce these SDP formulations?

Bounds through Semidefinite Programming

Method 1 Reduce the number of constraints and blocks by combining constraints.

$$\max_{Q \succeq 0} \min \left\{ 1 - \left\langle Q, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \right\rangle, -\left\langle Q, \begin{pmatrix} 1/6 & 1/3 \\ 1/3 & 1/6 \end{pmatrix} \right\rangle \right\},$$

Uses that the Ramsey multiplicity is invariant under color permutation. Purely combinatorial proof. *Strictly stronger than considering partitions (Balogh et al. 2017).*

Method 2 Reduce the number of variables by block diagonalization.

$$\max_{x,y \geq 0} \min \left\{ 1 - \frac{x}{2} - \frac{y}{2}, -\frac{x}{2} + \frac{y}{6} \right\}.$$

Particularly strong when combined with Method 1. Essentially an application of Schur's Lemma. Symmetries are easily determined combinatorially.

Generalizes the antiinvariant split of Razborov (2010). Similar to diagonalization in SOS literature (Gatermann and Parrilo 2004). See also Bachoc et al. (2012).

Bounds through Semidefinite Programming

Method 1 Reduce the number of constraints and blocks by combining constraints.

$$\max_{Q\succeq 0} \min \left\{1 - \left\langle Q, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \right\rangle, - \left\langle Q, \begin{pmatrix} 1/6 & 1/3 \\ 1/3 & 1/6 \end{pmatrix} \right\rangle \right\},$$

Uses that the Ramsey multiplicity is invariant under color permutation. Purely combinatorial proof. *Strictly stronger than considering partitions (Balogh et al. 2017).*

Method 2 Reduce the number of variables by block diagonalization.

$$\max_{x,y\geq 0} \min \left\{1 - \frac{x}{2} - \frac{y}{2}, -\frac{x}{2} + \frac{y}{6}\right\}.$$

Particularly strong when combined with Method 1. Essentially an application of Schur's Lemma. Symmetries are easily determined combinatorially.

Generalizes the antiinvariant split of Razborov (2010). Similar to diagonalization in SOS literature (Gatermann and Parrilo 2004). See also Bachoc et al. (2012).

Leveraging Symmetries

Theorem (Kiem, Pokutta, S. 2022+)

$m_{4,4} \geq 0.02961$ and $m_{5,5} \geq 0.001557$ from $N = 9$.

Theorem (Cummings et al. 2013)

$m_{3,3,3} = 1/25 = 1/(R_{3,3} - 1)^2$ and the only extremal constructions are based on $R_{3,3}$.

Theorem (Kiem, Pokutta, S. 2022+)

$m_{3,3,3,3} = 1/256 = 1/(R_{3,3,3} - 1)^2$ from $N = 6$.

Open Problem: $m_{3,\ldots,3} = (R_{3,\ldots,3} - 1)^{-2}$ for all c?

Leveraging Symmetries

Theorem (Kiem, Pokutta, S. 2022+)

$m_{4,4} \geq 0.02961$ and $m_{5,5} \geq 0.001557$ from $N = 9$.

Theorem (Cummings et al. 2013)

$m_{3,3,3} = 1/25 = 1/(R_{3,3} - 1)^2$ and the only extremal constructions are based on $R_{3,3}$.

Theorem (Kiem, Pokutta, S. 2022+)

$m_{3,3,3,3} = 1/256 = 1/(R_{3,3,3} - 1)^2$ from $N = 6$.

Open Problem: $m_{3,\dots,3} = (R_{3,\dots,3} - 1)^{-2}$ for all c?

Leveraging Symmetries

Theorem (Kiem, Pokutta, S. 2022+)

$m_{4,4} \geq 0.02961$ and $m_{5,5} \geq 0.001557$ from $N = 9$.

Theorem (Cummings et al. 2013)

$m_{3,3,3} = 1/25 = 1/(R_{3,3} - 1)^2$ and the only extremal constructions are based on $R_{3,3}$.

Theorem (Kiem, Pokutta, S. 2022+)

$m_{3,3,3,3} = 1/256 = 1/(R_{3,3,3} - 1)^2$ from $N = 6$.

Open Problem: $m_{3,\ldots,3} = (R_{3,\ldots,3} - 1)^{-2}$ for all c?

Leveraging Symmetries

Theorem (Kiem, Pokutta, S. 2022+)

$m_{4,4} \geq 0.02961$ and $m_{5,5} \geq 0.001557$ from $N = 9$.

Theorem (Cummings et al. 2013)

$m_{3,3,3} = 1/25 = 1/(R_{3,3} - 1)^2$ and the only extremal constructions are based on $R_{3,3}$.

Theorem (Kiem, Pokutta, S. 2022+)

$m_{3,3,3,3} \geq 1/256 - \varepsilon$ for some small ε from $N = 6$.

Open Problem: $m_{3,\dots,3} = (R_{3,\dots,3} - 1)^{-2}$ for all c?

Thank you for your attention!

Selected related literature

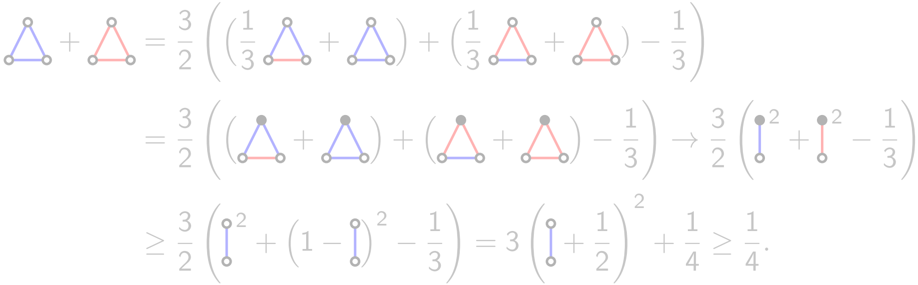
- Thomason, A. "Graph products and monochromatic multiplicities." Combinatorica 17.1 (1997): 125-134.
- Razborov, A. "Flag algebras." The Journal of Symbolic Logic 72.4 (2007): 1239-1282.
- Razborov, A. "On 3-hypergraphs with forbidden 4-vertex configurations." SIAM Journal on Discrete Mathematics 24.3 (2010): 946-963.
- Cummings, J., et al. "Monochromatic triangles in three-coloured graphs." Journal of Combinatorial Theory, Series B 103.4 (2013): 489-503.
- Balogh, J., et al. "Rainbow triangles in three-colored graphs." Journal of Combinatorial Theory, Series B 126 (2017): 83-113.
- Gatermann, J., and Parrilo, P.. "Symmetry groups, semidefinite programs, and sums of squares." Journal of Pure and Applied Algebra 192.1-3 (2004): 95-128.
- Bachoc, C., et al. "Invariant semidefinite programs." Handbook on semidefinite, conic and polynomial optimization. Springer, Boston, MA, 2012. 219-269.

Proof of Goodman's Result

An **upper bound** follows by considering the sequence of, e.g., (1) evenly-split complete bipartite graphs $K_{n/2,n/2}$ or (2) binomial random graphs $G(n, 1/2)$.

We saw: How to generalized the bipartite construction computationally.

A matching **lower bound** can symbolically be derived through



We saw: How to formalize and simplify this through Flag Algebras.

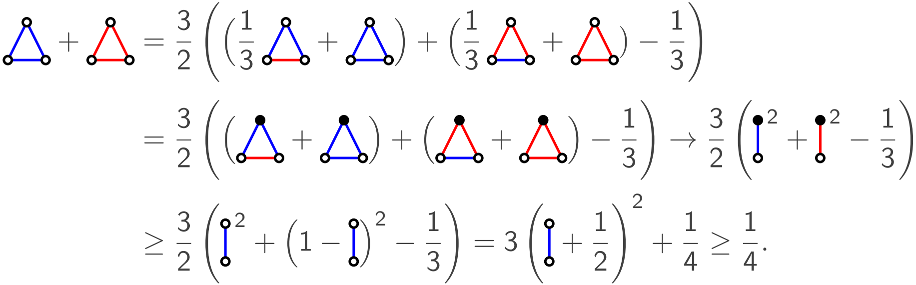
$\square$

Proof of Goodman's Result

An **upper bound** follows by considering the sequence of, e.g., (1) evenly-split complete bipartite graphs $K_{n/2,n/2}$ or (2) binomial random graphs $G(n,1/2)$.

We saw: How to generalized the bipartite construction computationally.

A matching **lower bound** can symbolically be derived through



We saw: How to formalize and simplify this through Flag Algebras. □