ZUSE INSTITUTE BERLIN

Towards Flag Algebras in Additive Combinatorics

CANT 2023 at CUNY

Juanjo Rué Perna and **Christoph Spiegel**

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Towards Flag Algebras in Additive Combinatorics

1. The Rado Multiplicity Problem
2. Lower bounds through Flag algebras in $\mathbb{F}_q^n$
3. Constructive upper bounds through blow-ups
4. Outlook

Definition of the problem

Given a **coloring** $\gamma : \mathbb{F}_q^n \to [c]$ and linear map L, we are interested in

$$\mathcal{S}_L(\gamma) \stackrel{\text{def}}{=} \{\mathbf{s} \in (\mathbb{F}_q^n)^m : L(\mathbf{s}) = \mathbf{0}, s_i \neq s_j \text{ for } i \neq j, \mathbf{s} \in \gamma^{-1}(\{i\})^m \text{ for some } i\}. \tag{1}$$

Rado (1933) tells us that $\mathcal{S}_L(\gamma) \neq \emptyset$ for large enough n if L satisfies *column condition*.

The **Rado Multiplicity Problem** is concerned with determining

$$m_{q,c}(L) \stackrel{\text{def}}{=} \lim_{n\to\infty} \min_{\gamma\in\Gamma(n)} |\mathcal{S}_L(\gamma)| \,/\, |\mathcal{S}_L(\mathbb{F}_q^n)|.$$

Limit exists by monotonicity and $0 < m_{q,c}(L) \leq 1$ if L is partition regular. L is c-**common** if $m_{q,c}(L) = c^{1-m}$ (the value attained in a uniform random coloring).

This reminds us of an old question of Erdős in graph theory ...

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Previous results

- Graham et al. (1996) gave lower bound for **Schur triples** in 2-colorings of $[n]$, later independently resolved by Robertson and Zeilberger / Schoen / Datskovsky.
- Cameron et al. (2007) showed that the nr. of solutions for linear equations with **an odd nr. of variables** only depends on cardinalities of the two color classes.
- Parrilo, Robertson and Saracino (2008) established bounds for the minimum number of **monochromatic** 3-**APs** in 2-colorings of $[n]$ (not 2-common in $\mathbb{N}$).
- For $r = 1$ and m even, Saad and Wolf (2017) showed that any 'pair-partitionable' L is 2-common in $\mathbb{F}_q^n$. Fox, Pham, and Zhao (2021) showed that this is necessary.
- Kamčev et al. (2021) characterized some non-common L in $\mathbb{F}_q^n$ with $r > 1$.
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What can we contribute?

We are interested in particular L and fixed q. Analogous to e.g. determining Ramsey Multiplicity of K_4 for 2-uniform graphs (has a long history starting with Erdős in 1962).

Theorem (Rué and S., 2023)

We have $1/10 < m_{q=5,c=2}(L_{4\text{-}AP}) \leq 0.1\overline{031746}$.

Saad and Wolf (2017) previously established an u.b. of 0.1247 with no no-trivial l.b. known.

Proposition (Rué and S., 2023)

We have $m_{q=3,c=3}(L_{3\text{-}AP}) = 1/27$.

Similar to Cummings et al. (2013) extending a result of Goodman (1959) about triangles.

Proofs are computational and applicable to other questions:

- **Upper bounds** through (iterated) blow-up constructions of particular finite colorings.
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An improvement on a trivial lower bound

The parameter $s_L(\gamma) \stackrel{\text{def}}{=} |\mathcal{S}_L(\gamma)| \,/\, |\mathcal{S}_L(\mathbb{F}_q^n)|$ satisfies the averaging equality

$$s_L(\gamma) = \sum_{\delta\in\Gamma(k)} p(\delta,\gamma)\, s_L(\delta) + o(1) = \mathbb{E}^{(\gamma)}_{\delta\in\Gamma(k)} s_L(\delta) + o(1) \tag{2}$$

once k is large enough. This implies an immediate trivial lower bound of

$$m_{q,c}(L) \geq \min_{\delta\in\Gamma(k)} s_L(\delta). \tag{3}$$

If we *magically* found some coefficients a_δ satisfying $\mathbb{E}^{(\gamma)}_{\delta\in\Gamma(k)} a_\delta = o(1)$, we would get

$$m_{q,c}(L) \geq \min_{\delta\in\Gamma(k)} s_L(\delta) - a_\delta. \tag{4}$$

But how would we find such a_δ? **Flag Algebras and Semidefinite Programming!**

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But how would we find such a_δ? **Flag Algebras and Semidefinite Programming!**

The SDP approach

Definition

The *flag algebra* (of the empty type) $\mathcal{A}$ is given by considering $\mathbb{R}\Gamma$, factoring out relations given by the averaging equality and defining an appropriate product.

There exists an element $C_L \in \mathcal{A}$ capturing the behavior of s_L and **the *semantic cone***

$$\mathcal{S} = \{f \in \mathcal{A} : \phi(f) \geq 0 \text{ for all } \phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})\} \quad (5)$$

captures those algebraic expressions corresponding to density expressions that are 'true'. We can therefore establish a lower bound for s_L by showing that

$$C_L - \lambda - \sum_{i=1}^{k} (f_i)^2 \in \mathcal{S} \quad \Rightarrow \quad C_L - \lambda \in \mathcal{S}. \quad (6)$$

Here $p(\sum_{i=1}^{k}(f_i)^2, \delta)$ corresponds to the a_δ on the previous slide! Such sum-of-squares (SOS) expressions are related to easily solvable Semidefinite Programs (SDPs).

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Challenges

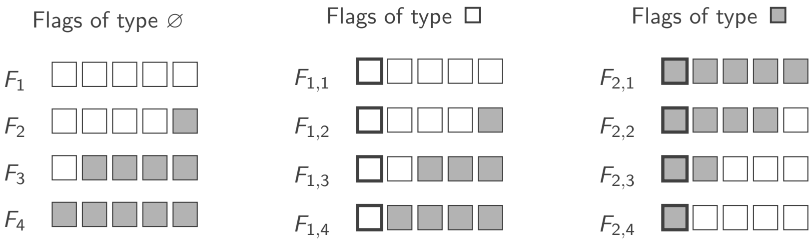
- Need an appropriate notion of density, isomorphism, and ‘partially fixed coloring’ both to (i) handle invariance and non-invariance and (ii) define different algebras.
- Solutions as defined previously do *not* satisfy an exact averaging equality. Need to introduce *fully dimensional solutions*, which asymptotically make up all solutions.
- Need to adequately solve isomorphisms problem from a practical perspective.
- (Almost) all SDP solvers work numerically, but we need algebraic expressions.

Lower bound of the Proposition

$m_{5,2}(L_{4\text{-AP}}) > 1/10$ follows by verifying that over all 3324 2-colorings of $\mathbb{F}_5^2$ we have

$$F_1 + F_4 + (F_2 + F_3)/5 - 1/10 \geq \sum_{i=1}^{2} \Big(9/10 \cdot \big[\!\big[(F_{i,1} + (5\,F_{i,2} - 5\,F_{i,3} - 10\,F_{i,4})/27)^2\big]\!\big]_{-1}$$
$$\ldots + 61/162 \cdot \big[\!\big[((F_{i,3} - F_{i,2})/2 + F_{i,4})^2\big]\!\big]_{-1}\Big),$$

and by noting that $F_{1,1} + F_{2,1} > 0$. Here the relevant flags F_i and $F_{i,j}$ are

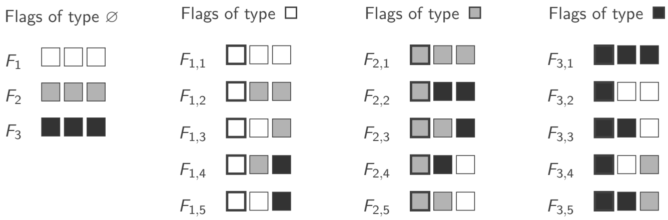


Lower bound of the Theorem

$m_{3,3}(L_{3\text{-AP}}) \geq 1/27$ follows by verifying that over all all 140 3-colorings of $\mathbb{F}_3^2$ we have

$$F_i - 1/27 \geq 26/27 \cdot \llbracket (F_{i,1} - 99/182\, F_{i,2} + 75/208\, F_{i,3} - 11/28\, F_{i,4} - 3/26\, F_{i,5})^2 \rrbracket_{-1}$$
$$\ldots + 1685/1911 \cdot \llbracket (F_{i,2} - 231/26960\, F_{i,3} + 1703/6740\, F_{i,4} - 1869/3370\, F_{i,5})^2 \rrbracket_{-1}$$
$$\ldots + 71779/431360 \cdot \llbracket (F_{i,3} - 358196/502453\, F_{i,4} - 412904/502453\, F_{i,5})^2 \rrbracket_{-1}$$
$$\ldots + 5431408/10551513 \cdot \llbracket (F_{i,4} - 1/4\, F_{i,5})^2 \rrbracket_{-1}$$

for any $i \in \{1, 2, 3\}$. Here the relevant flags F_i and $F_{i,j}$ are



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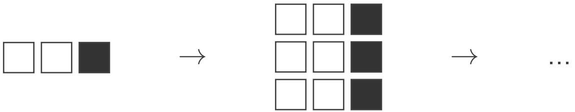
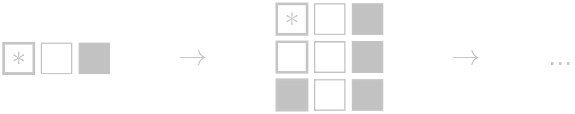
How to blow up colorings

We can *blow up* an colorings into a sequence of colorings with n tending to infinity.



Computing the density of solutions in the limit of this sequence is easy: simply check *not-necessarily-injective* subcolorings in the base construction. **This gives us an immediate upper bound from *any* coloring we can come up with ...**

In some cases we have a *free element* in which we can iterate the blowup-construction.



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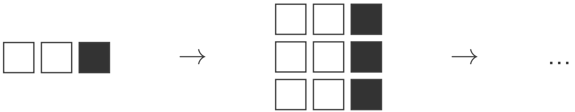

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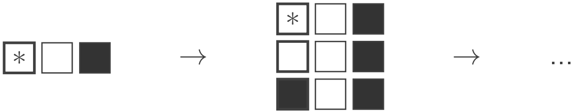
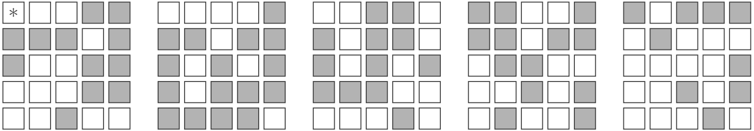


Proofs of the upper bounds

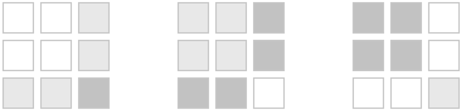
Upper bound of the Proposition

$m_{5,2}(L_{4\text{-AP}}) \leq 13/126$ follows from the iterated blow-up of this 2-coloring of $\mathbb{F}_5^3$:



Upper bound of the Theorem

$m_{3,3}(L_{3\text{-AP}}) \leq 1/27$ follows from the blow-up of this 3-coloring of $\mathbb{F}_3^3$:



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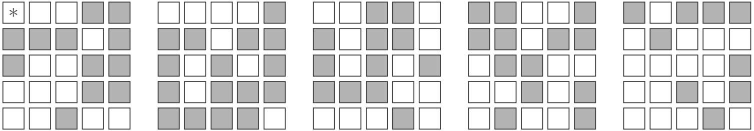

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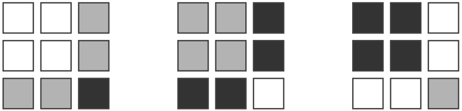

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Final Remarks

- Often one can extract stability results from Flag Algebra certificates.
- Steep computational hurdle: underlying structures grow exponentially (instead of quadratically for graphs or cubic for 3-uniform hypergraphs)

q/n	1	2	3	4	5
2	3	5	10	32	382
3	4	14	1028		
4	8	1648			
5	6	**3324**			

q/n	1	2	3	4
2	6	15	60	996
3	10	**140**	25 665 178	
4	30	1 630 868		
5	24	70 793 574		

Table: Number of 2- and 3-colorings of $\mathbb{F}_q^n$.

- No neat notion of subspaces makes generalizing to other groups difficult.

Code is available at `github.com/FordUniver/rs_radomult_23`

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Thank you for your attention!