Computational Flag Algebras in Lean

Flags in the Mountains | CSU Mountain Campus

Christoph Spiegel | Zuse Institute Berlin

2 June 2026

Two Caveats Up Front

1. **A talk about verifying results, not about obtaining them.**

   Not about a specific extremal combinatorial problem and not how one *solves* flag algebra problems. It is about how we can trust a flag algebra proof once we have one.

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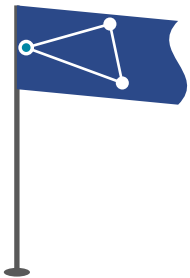
2. **The word ‘type’ is overloaded.**
   - In **Flag Algebras**: the fixed, labeled graph core shared by the family of all flags of that ‘type’.
   - In **Lean**: what kind of value a piece of data holds and the foundational notion of the logic itself.

To keep them apart, we call the labeled core of a flag its pole.



Flag Algebras, Certificates, and Trust

Our Running Example: Goodman's Bound

Goodman (1959)

$\liminf_{n\to\infty} d(K_3, H_n) + d(I_3, H_n) \geq 1/4$ for every sequence of graphs $(H_n)_{n\in\mathbb{N}}$.

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What do Flag Algebras actually give us?

1. **Notation.** Drop the redundant '$d(\cdot, H_n)$' and 'for every sequence of graphs (H_n)' throughout; just write $K_3 + I_3 \geq 1/4$ — *a real vector space spanned by graphs*.

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3. **Chain rule.** Re-express densities at higher order, collapsing parts of the algebra.
4. **Poles.** Each way of fixing finitely many labeled vertices yields its own algebra with the pole as unit, related to the others by order-preserving maps.

A Flag Algebra Certificate

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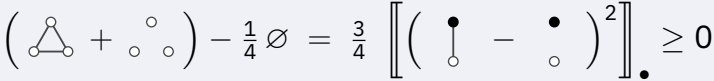
Combining all four, someone can hand you a certificate for this statement:



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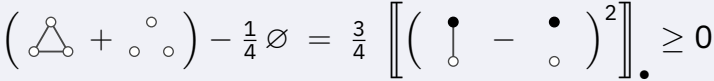

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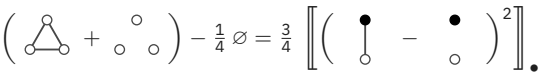
Crucially, finding the certificate should be the hard part. Checking a certificate should not: it has to be a purely mechanical computation that runs on modest hardware.*

* ideally

A Mechanical Verification of a Flag Algebra Certificate



The claimed certificate

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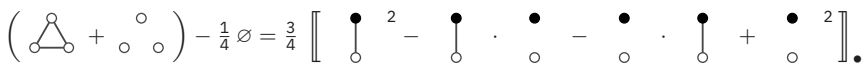

Apply linearity of the product

A Mechanical Verification of a Flag Algebra Certificate

Resolve all products (and canonizing graphs)

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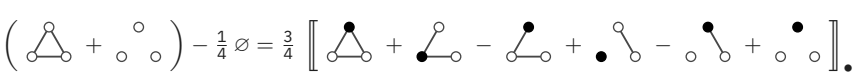

Canonize through commutativity and distributivity

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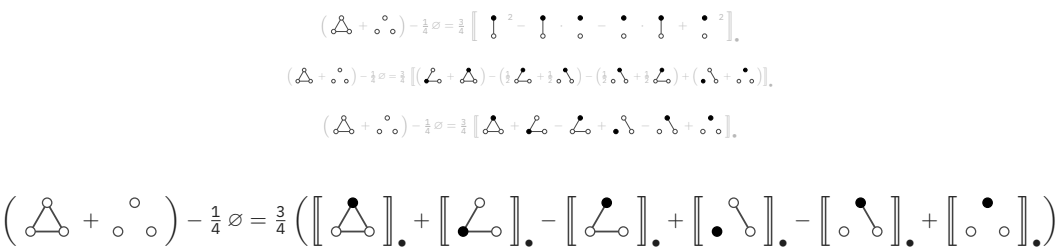

Apply linearity of the downward operator

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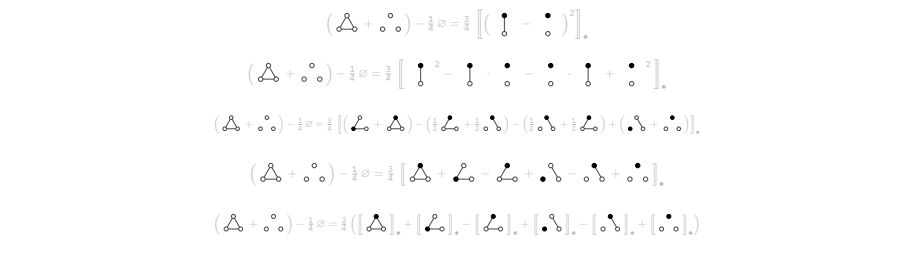

Apply downward operators

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Bring all terms to the smallest common order

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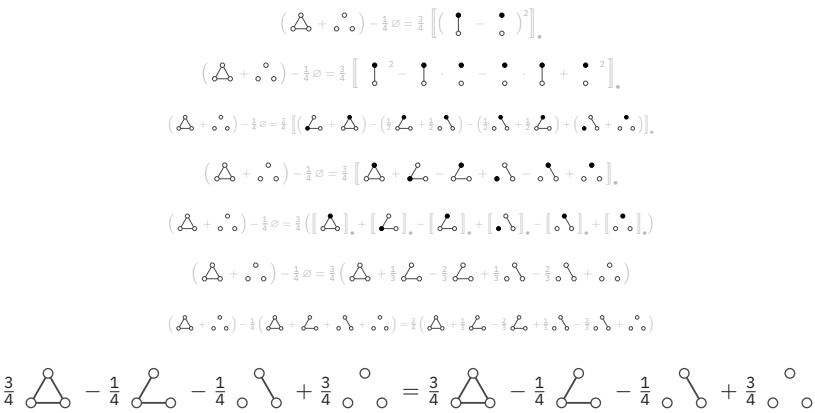

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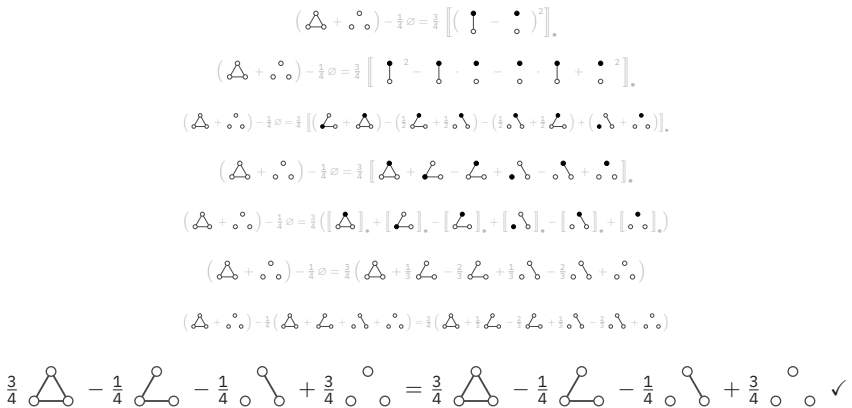

Compare coefficients

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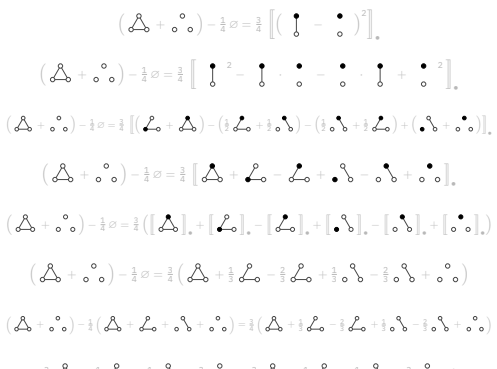

These proofs and their coefficients quickly grow far too large to carry out by hand...

Flag Algebra Certificates in the Wild

… so in practice everyone provides their own code that verifies the certificate for you.

Framework	Active	Structures	Language	Serialization	Certificate
Flagmatic E. Vaughan	2012–14	graphs, 3-graphs, oriented	Python/Cython	JSON	rational PSD
permpack J. Sliačan	2016–18	permutations	SageMath	—	—
Flagmatic (Sage fork) J. Sliačan	2018–22	graphs, 3-graphs, oriented	Python/Cython	JSON	rational PSD
flag-algebra Coregliano, Parente, Sato	2018	tournaments, graphs, digraphs	C++	plain text	PSD / SOS
FlagSOS.jl D. Brosch	2020–25	(hyper)graphs, codes, trees	Julia	JLD2	SOS (rational)
rust-flag-algebra R. de Verclos	2020–25	graphs, digraphs, colored	Rust	SDPA / HTML	PSD (numeric)
FlagAlgebraToolbox L. Bodnár	2025–26	(hyper)graph (coloring)s, digraphs	SageMath	—	rational PSD
C. Spiegel *et al.*	2023–24	graph colorings	Python/Sage	YAML	rational PSD
B. Lidický *et al.*	2014–25	graphs, hypergraphs, …	C++ / Sage	`.dat-s` + `.sage`	per paper
D. Brosch *et al.*	2024–25	rooted trees	Julia	JLD2	SOS (rational)
D. Brosch *et al.*	2025	graph colorings	C++ / Julia	—	SDP + SAT/ILP

What Are We Actually Trusting?

What we put our trust in

- Razborov's Flag Algebra Theory
  product, averaging, Positivstellensatz, ...
- The Bespoke Verification Code
  flag product, downward operator, certificate parsing, ...
- Graph and Combinatorics Libraries
  canonical labeling (`nauty`, Bliss, Traces), isomorph-free generation (`geng`), SageMath, NetworkX, igraph, ...
- Exact Arithmetic
  GMP, FLINT, numpy, LAPACK, ...
- Compiler, Runtime & Toolchain
  GCC, Clang, LLVM, CPython, Julia JIT, ...

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  by other groups, in other languages
- Trusted Dependencies
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- Integration Test Suites
  against known values, small cases, OEIS, ...
- 'Safe' Languages
  memory and type safety

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The ultimate 'safe' language is one whose type system also checks the mathematics itself.

Formal Proof Verification and Verified Computing in Lean

Two Views of Formal Proof Verification in Lean

1. A Playground for Axiomatic Logic.

You prove and the machine checks down to the axioms. What you end up trusting:

- The playground itself and the usual software / hardware stack.
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▷ *A formally verified graph theoretic statement building on* `mathlib` *notions.*

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- ▷ *Verifying the handshaking lemma computationally for concrete graphs.*

Curry–Howard Correspondence and What We No Longer Trust

Curry–Howard: propositions *are* types and proofs *are* programs.

The handshaking lemma we *stated* and the one we *ran* are the same object: the very notions `mathlib` uses to *state* mathematics can be used for computations.

Caveat: `Classical` reasoning yields proofs with nothing to *run* — and `mathlib` uses `Classical` and `noncomputable` *a lot*, as we already saw in our trivial example. But it is avoidable!

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- The used dependencies
- The paper-to-code correspondence

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- Lean's kernel and toolchain
- The axioms it builds on
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▷ *Let's see computational flag algebras in Lean!*

Wrapping Up

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Code not yet public, hopefully will have a cleaned up version ready by end of year...

Thank you!