

An unsure talk on an un-Schur problem

Flags in the Mountains | CSU Mountain Campus

Olaf Parczyk | **Christoph Spiegel** | Zuse Institute Berlin

5 June 2026

Why this problem?

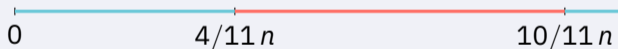
Monochromatic Schur triples and Ramsey properties

Schur (1916)

Every c -coloring of $[n]$ contains a monochromatic Schur triple for $n \geq n_0(c)$.

Asymptotically a positive proportion must be monochromatic; Graham, Rödl, and Ruciński (1996) showed ≥ 0.04 for $c = 2$ via a result of Goodman (1959).

Question (Chen and Graham, SOCA 96, \$100). Is Zeilberger's construction giving $2/11 = 0.\overline{18}$ optimal?



Datskovsky 2003; Schoen 1999; Robertson and Zeilberger 1998

Yes, and the construction is unique!

Rainbow Schur triples and anti-Ramsey properties

Alekseev and Savchev 1987; Schönheim 1990

Every 3-coloring of $[n]$ must contain an un-Schur triple, i.e. a **rainbow** Schur triple, for large enough n as long as each color class covers at least a $1/4$ proportion of $[n]$.

Question. What is the maximum proportion of Schur triples that can be rainbow?

This (surprisingly) seems to not have been explicitly asked before, even though the graph-theory equivalent due to Erdős and Sós was resolved by Balogh et al. (2017).

Parczyk and S. (2024+)

At least 0.4 and at most 0.66656. We conjecture the lower bound to be tight.

Constructive lower bounds

Two obvious candidates

Intervals. $c_{\text{int}} : i \mapsto \begin{cases} \text{blue} & \text{if } 1 \leq i < n/3 \\ \text{red} & \text{if } n/3 \leq i < 2n/3 \\ \text{green} & \text{if } 2n/3 \leq i \leq n \end{cases}$ gives $2/9 = 0.\overline{2}$.



Modulus. $c_{\text{mod}} : i \mapsto \begin{cases} \text{red} & \text{if } i \equiv 1 \pmod{4} \\ \text{blue} & \text{if } i \equiv 2 \pmod{4} \\ \text{green} & \text{if } i \equiv 3 \pmod{4} \\ \text{blue} & \text{if } i \equiv 0 \pmod{4} \end{cases}$ gives $3/8 = 0.375$.



The best construction

However, the best construction gives $2/5 = 0.4$ by combining both ideas:

$$c^* : i \mapsto \begin{cases} \text{blue} & \text{if } i \text{ is odd and } i \leq 2n/5 \\ \text{red} & \text{if } i \text{ is odd and } i > 2n/5 \\ \text{green} & \text{if } i \text{ is even} \end{cases} .$$



It was found by looking at the (unique) best solutions for $n = 10, 11, 12, \dots, 30$ obtained by solving a MILP formulation of the underlying problem.

An upper bound

An upper bound, via rainbow triangles

A Schur triple is an ordered (x, y, z) with $x + y = z$; write $r_c(z)$ for the number of them that are rainbow with top element z . We bound $\sum_z r_c(z)$, against the $\binom{n}{2}$ Schur triples in total.

The idea. Triangles in K_{n+1} map to Schur triples via

$$f : (v_1, v_2, v_3) \mapsto (v_2 - v_1, v_3 - v_2, v_3 - v_1),$$

under the edge-coloring $e \mapsto c(\max e - \min e)$. A triple with top z has exactly $n + 1 - z$ triangle preimages, each rainbow iff the triple is, so Balogh et al. (2017) gives

$$\sum_{z=1}^n r_c(z) (n + 1 - z) \leq \left(\frac{1}{15} + o(1)\right) n^3.$$

A reweighting argument strips the weight $(n + 1 - z)$:
 $\sum_z r_c(z) \leq 0.333 n^2$, i.e. at most a **0.667 proportion** of all $\binom{n}{2}$ Schur triples.

Concluding remarks

What about arithmetic progressions?

Related question 1. What is the maximum proportion of k -term arithmetic progressions that can be rainbow? The case $k = 3$ was remarked to be $2/3$ by Jungić et al. (2003). This extends to: the maximum fraction is at least

$$\prod_{i=1}^m (1 - 1/p_i) \quad \text{if } k = p_1^{a_1} \cdots p_m^{a_m},$$

by coloring mod $p_1 \cdots p_m$, and at most $(1 - 1/k)$. We again conjecture the lower bound to be tight.

Related question 2. One can also consider similar questions in \mathbb{Z}_n instead of $[n]$, which made the problem easier for monochromatic Schur triples, though here the optimal construction seems to depend on the precise prime decomposition of n .

Thank you!