The four-color Ramsey Multiplicity of Triangles

LIMDA Seminar, UPC Barcelona

Christoph Spiegel (Zuse Institute Berlin)

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Ongoing work with ...

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ZIB / TU Berlin

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ZIB / TU Berlin

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The four-color Ramsey Multiplicity of Triangles

The Ramsey Multiplicity Problem

Theorem (Ramsey 1930 – Multicolor Version)

For any $t_1, \ldots, t_c \in \mathbb{N}$ there exists $R_{t_1,\ldots,t_c} \in \mathbb{N}$ s.t. any c-edge-coloring of K_n with $n \geq R_{t_1,\ldots,t_c} \in \mathbb{N}$ contains an clique of size t_i with edges colored i for some $1 \leq i \leq c$.

A well-known question

Can we determine $R_{t_1,\ldots,t_c}$?

A related question

How many cliques are required?

Theorem (Goodman 1959 – Asymptotic Version)

Asymptotically at least $1/4$ of all triangles are monochromatic in any 2-edge-coloring.

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Beyond Goodman's Result

Notation. Let $\mathcal{G}_n = \{G : E(K_n) \to [c]\}$ denote all c-edge-colorings of K_n, G_i the subgraph of K_n given by color i and $k_{t_i}(G_i)$ the fraction of t_i-cliques in G_i.

Problem (Ramsey Multiplicity)

What is the value of $m_{t_1,\ldots,t_c} = \lim_n \min_{G \in \mathcal{G}_n} k_{t_1}(G_1) + \ldots + k_{t_c}(G_c)$?

The success of the binomial random graph for $m_{3,3}$ lead to the following conjecture.

Conjecture (Erdős 1962)

$m_{t,t} = 2^{1-\binom{t}{2}}$ for any $t \geq 2$. **False for $t \geq 4$ (Thomason 1989)**

Determining even $m_{4,4}$ is still an open and very hard problem... But what if we only consider triangles and increase the number of colors?

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Theorem (Goodman 1959 – Asymptotic Version)

$m_{3,3} = 1/4$.

Besides random graphs, a matching upper bound is for example also given by complete bipartite graphs, i.e., the blowup of R_3-coloring (the ‘one-color’ Ramsey number).

Theorem (Cummings et al. 2013)

$m_{3,3,3} = 1/25$ and all extremal sequences are based on blowups of the $R_{3,3}$-coloring.

Using either of the two $R_{3,3,3}$-colorings, one has $m_{3,3,3,3} \leq 1/256$.

Theorem (Kiem, Pokutta, S. 2023+)

$m_{3,3,3,3} \geq 1/256 - \varepsilon$ for some small ε.

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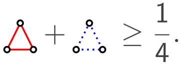
Goodman's original proof

We want to show that



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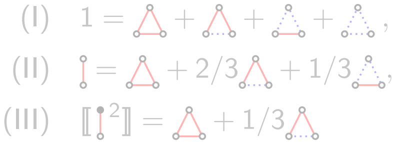

Equation (3) in Goodman's paper

Equation (4) in Goodman's paper,

where  and $[\![\cdot]\!]$ is the *downward operator*. (I) − 3(II) + 3(III) gives

where we used  (CS) for the first inequality. □

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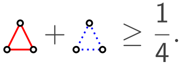

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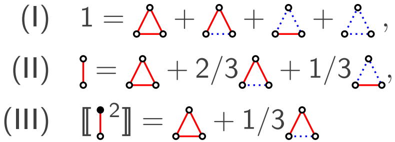

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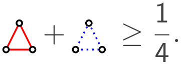

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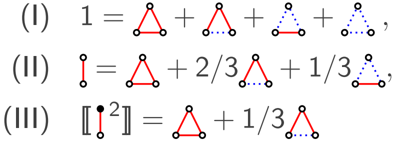

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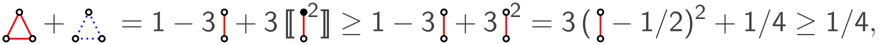
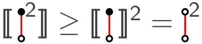
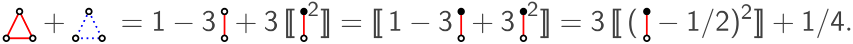


where we used  (CS) for the first inequality. $\square$

Rephrasing Goodman's proof

Instead of applying CS, we could embrace the downward operator through



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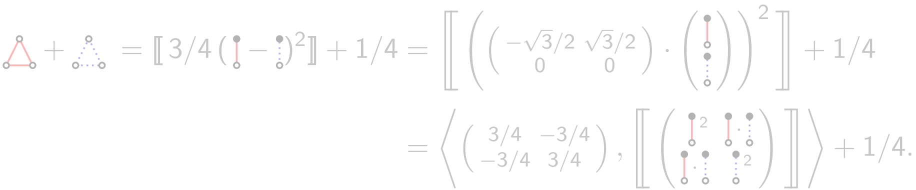

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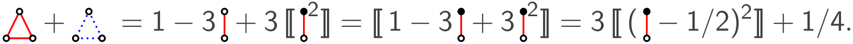

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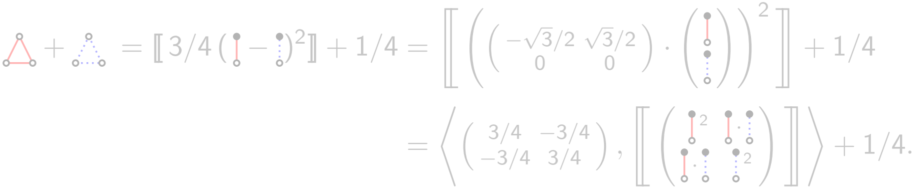

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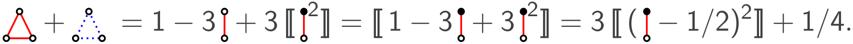

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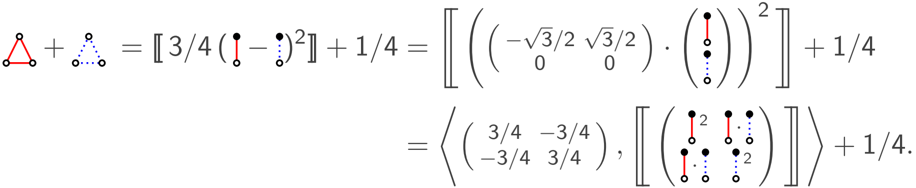

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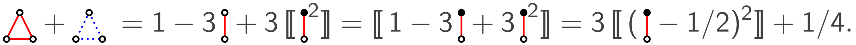

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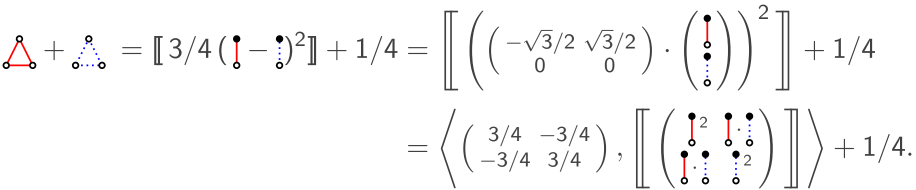

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Flag Algebras

Notation. A *type* τ is a fully labelled coloring and a *flag* $F \in \mathcal{F}^\tau$ of type τ is a coloring $\downarrow F$ with partial labels inducing τ. Write $\mathcal{F}_n^\tau$ for flags of order n and note that $\mathcal{G}_n = \mathcal{F}_n^\varnothing$.

Definition (The Flag Algebra of type τ; Razborov 2007)

The *flag algebra* $\mathcal{A}^\tau$ is given by considering $\mathbb{R}\mathcal{F}^\tau/\mathcal{K}$ for

$$\mathcal{K} = \{F - \sum_{F' \in \mathcal{F}_n^\tau} p(F; F')F' : F \in \mathcal{F}^\tau, n \geq v(F)\}$$

and defining the product

$$F_1 \cdot F_2 = \sum_{F' \in \mathcal{F}_n^\tau} p(F_1, F_2; F')\, F' \quad \text{for any } n \geq v(F_1) + v(F_2) - v(\tau).$$

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The Semantic Cone of Flag Algebras

Any *convergent* sequence $G_n \in \mathcal{G}$ defines a *limit functional* $p(F, G_n) \to \varphi(F)$ on $\mathcal{A}^{\varnothing}$.

Theorem (Razborov 2007)

φ is a limit functional if and only if $\varphi \in Hom^{+}(\mathcal{A}, \mathbb{R}) = \{\varphi \in Hom(\mathcal{A}, \mathbb{R}) : \varphi|_{\mathcal{G}} \equiv 0\}$.

We can phrase our problem of minimizing $F_0 =$  $+$  as the optimization problem

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Directly optimizing over the semantic cone is hard, but we can use SOS through

$$\max_{Q \succeq 0} \min_{G \in \mathcal{G}_N} d(F_0; G) - \sum_{\tau} \left\langle Q, \left([\![d(F_1, F_2; G)]\!]_{\tau} \right)_{F_1, F_2 \in \mathcal{F}_f^{\tau}} \right\rangle$$

where $0 \leq v(\tau) \leq N - 2$ and $f = \lfloor (N - v(\tau))/2 \rfloor$.

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Leveraging Symmetries

Goodman's result relies on computations on colorings of order $N = 3$:

$$\max_{Q \succeq 0} \min \left\{ 1 - \left\langle Q, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle, - \left\langle Q, \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix} \right\rangle, - \left\langle Q, \begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix} \right\rangle, 1 - \left\langle Q, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \right\} = 1/4.$$

Increasing N usually both improves the bound and makes the SDP harder to solve.

N	*value*	*time*	*memory*
6	0.02875	0.2s ±0.0	81.2MB ±24.7
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Table: Complexity of SDP problem formulations for $m_{4,4}$ using CSDP

How can we use combinatorial information to reduce these SDP formulations?

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Method 1 Reduce the number of constraints and blocks by combining constraints.

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Strictly stronger than considering partitions (Balogh et al. 2017).

Method 2 Reduce the number of variables by block diagonalization.

$$\max_{x,y \geq 0} \min \left\{ 1 - \frac{x}{2} - \frac{y}{2}, -\frac{x}{2} + \frac{y}{6} \right\}.$$

Generalizes the antiinvariant split of Razborov (2010). Similar to diagonalization in SOS literature (Gatermann and Parrilo 2004). See also Bachoc et al. (2012).

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Leveraging Symmetries

We derived our result with $N = 6$ vertices, giving a 3GB+ SDP with 130k variables and 120k constraints that **takes a day to solve numerically**.

Challenge: Turn the numerical solution into rigorous proof. Getting a small ε below the bound is easy (round the LDL-decomposition) but hitting the exact value requires formulating and solving an appropriate exact Linear Program (LP).

We also derived new (non-tight) lower bounds for $m_{4,4}$ and $m_{5,5}$ using 2-colorings on $N = 9$ vertices, where numerically solving the SDP takes **weeks**.

Open Problem: $m_{3,\dots,3} = (R_{3,\dots,3} - 1)^{-2}$ for all c?

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Challenge: Turn the numerical solution into rigorous proof. Getting a small ε below the bound is easy (round the LDL-decomposition) but hitting the exact value requires formulating and solving an appropriate exact Linear Program (LP).

We also derived new (non-tight) lower bounds for $m_{4,4}$ and $m_{5,5}$ using 2-colorings on $N = 9$ vertices, where numerically solving the SDP takes **weeks**.

Open Problem: $m_{3,\dots,3} = (R_{3,\dots,3} - 1)^{-2}$ for all c?

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Thank you for your attention!

Selected related literature

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