New Ramsey Multiplicity Bounds and Search Heuristics

Discrete Mathematics Days 2022

Christoph Spiegel

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Results are joint work with...

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Definitions and upper bounds

Letting $k_t(G)$ denote the fraction of all possible t-cliques in G, we are interested in

$$c_t(n) = \min\{k_t(\overline{G}) + k_t(G) : |G| = n\}.$$

Theorem (Ramsey 1930)

For every $t \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $c_t(n) > 0$ iff $n \geq n_0$.

Let us write $c_t = \lim_{n\to\infty} c_t(n)$. Goodman showed that $c_3 = 1/4$ in 1959. Erdős conjectured that $c_t = 2^{1-\binom{t}{2}}$ in 1962 and this was extended by Burr and Rosta.

Theorem (Thomason 1989)

$c_4 \leq 0.976 \cdot 2^{-5}$, $c_5 \leq 0.906 \cdot 2^{-9}$, *and* $c_t \leq 0.936 \cdot 2^{1-\binom{t}{2}}$ *for* $t \geq 6$.

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Theorem (Conlon 2011)

$c_t \geq 2.18^{-t^2(1+o(1))}$ *for any* $t \geq 4$, *implying* $c_t \cdot 2^{\binom{t}{2}-1} = \Omega(2^{-0.62t^2})$.

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Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld/Nieß '11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

We can show the first improved upper bounds in 27 years:

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

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An off-diagonal variant

Question. Determining c_3 is easy, but even c_4 has been unresolved for over 60 years, so can we say more when studying the off-diagonal variant

$$c_{s,t} = \lim_{n\to\infty} \min\{k_s(\overline{G}) + k_t(G) : |G| = n\}?$$

A famous result of Reiher from 2016 implies that $c_{2,t} = 1/(t-1)$.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_{3,4} = 689 \cdot 3^{-8}$ and any large enough graph G admits a strong homomorphism into the Schläfli graph after changing at most $O(k_3(\overline{G}) + k_4(G) - c_{3,4})\, v(G)^2$ edges.

We can also show that $c_{3,5} = 24011 \cdot 3^{-12}$ and $0.00768 < c_{4,5} < 0.00794$.

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Graph blow-ups and strong homomorphisms

Definition

The *m-fold blow-up* $C[m]$ of a graph C is given by replacing each vertex in C with an independent set of size m. Two vertices are adjacent if the originals were.

We can derive an upper bound for $c_{s,t}$ from *any* graph C through

$$c_{s,t} \leq \lim_{m\to\infty} \left(k_s(\overline{C[m]}) + k_t(C[m])\right). \tag{1}$$

This discrete optimization problem is amenable to computational tools since the fraction of maps $G \to C$ forming strong homomorphisms is invariant under blow-up.

Proposition (Thomason 1987)

$\lim_{m\to\infty} k_t(C[m]) = \frac{n^{\underline{t}}\, k_t(C)}{n^t}$ *and* $\lim_{m\to\infty} k_t(\overline{C[m]}) = \sum_{j=1}^{t} \frac{S(t,j)\, n^{\underline{j}}\, k_j(\overline{C})}{n^t}$.

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Constructing graphs through search heuristics

For $\mathbf{s} \in \{0,1\}^{\binom{n}{2}}$ let $C_\mathbf{s} = \left([n], \{ij : i < j, s_{\binom{j-1}{2}+i} = 1\}\right)$ and consider

$$\min_{\mathbf{s}\in\{0,1\}^{\binom{n}{2}}} \sum_{j=1}^{s} \frac{S(t,j)\, n^{\underline{j}}\, k_j(\overline{C_\mathbf{s}})}{n^t} + \frac{n^{\underline{t}}\, k_t(C_\mathbf{s})}{n^t}. \tag{2}$$

Many approaches to solve this optimization problem exist:

Approach 1. For $n \lessapprox 7$ we can check all states $\mathbf{s}$ exhaustively.

Pros: Easy to implement. *Cons:* Quickly succumbs to combinatorial chaos.

The most established search heuristics are modifications of Hill Climbing: *Simmulated Annealing* by Kirkpatrick, Gelatt and Vecchi (1983) and *Tabu search* by Glover (1986).

Running these with $s = 3$, $t = 4$ and $n = 27$ yields the Schläfli graph for $c_{3,4}$.

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Approach 2. For $n \lessapprox 10$ we can generate all graphs up to isomorphism.

Pros: Implementations already exists. *Cons:* Still considers graphs far from optimal.

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Approach 3. For $n \lessapprox 15$ we can use a Bounded Search Tree.

Pros: Ignores unimportant graphs. *Cons:* Can be tricky to implement efficiently.

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Many approaches to solve this optimization problem exist:

Approach 4. For $n \lessapprox 40$ we can use Search Heuristics.

Cons: No guarantees. *Pros:* Easy to implement, fast and (often) accurate in practice!

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Constructing Cayley graphs through search heuristics

Running the graph search with $s = t = 4$ yields little of interest up to $n = 40$.

Thomason's constructions are based on computing the values of XOR-graph-products. The results are in fact Cayley graphs in $C_3^{\times 2} \times C_2^{\times 5}$ and $C_3 \times C_2^{\times 6}$.

Idea. Why not directly search Cayley graph constructions?

Given a group G, define $S = \{\{s, s^{-1}\} : s \in G^\star\}$ and let $\mathbf{s} \in \{0,1\}^{|S|}$ represent the generating set of the Cayley graph $C_\mathbf{s} = \left(G, \{g_1 g_2 : g_1 s = g_2 \text{ for some } \{s, s^{-1}\} \in S\}\right)$.

Since $|G|/2 < |S| < |G|$ the number of variables is now linear in the number of vertices!

Running searches with $s = t = 4$ and $G = C_3 \times C_2^{\times 8}$ as well as $s = t = 5$ and $G = C_3 \times C_2^{\times 6}$ gives the improved upper bounds for c_4 and c_5.

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Since $|G|/2 < |S| < |G|$ the number of variables is now linear in the number of vertices!

Running searches with $s = t = 4$ and $G = C_3 \times C_2^{\times 8}$ as well as $s = t = 5$ and $G = C_3 \times C_2^{\times 6}$ gives the improved upper bounds for c_4 and c_5.

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1. The Ramsey Multiplicity Problem

2. Search Heuristics for Upper Bounds

3. Some Context and Related Problems

Understanding the full clique trade-off

Consider the region $\Omega_{s,t} \subseteq [0,1]^2$ of points (x, y) for which there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs of increasing order satisfying $k_s(\overline{G_n}) \to x$ and $k_t(G_n) \to y$.

Proposition (Parczyk, Pokutta, S., and Szabó 2022+)

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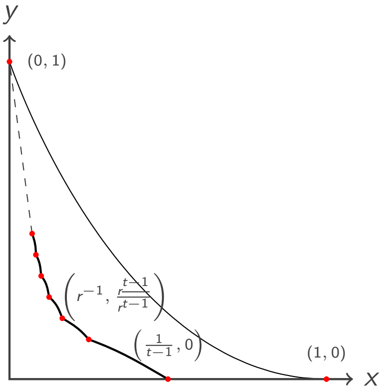

Figure: The region $\Omega_{2,t}$ due to Razborov (2008), Nikiforov (2011), and Reiher (2016).

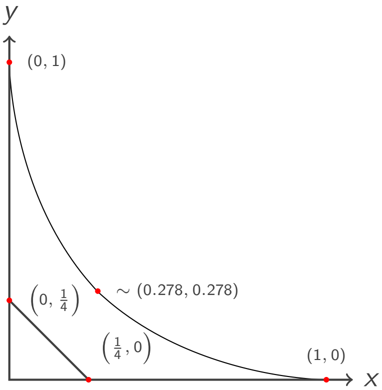

Figure: The region $\Omega_{3,3}$ due to Goodman (1959) as well as Huang, Linial, Naves, Peled, and Sudakov (2014).

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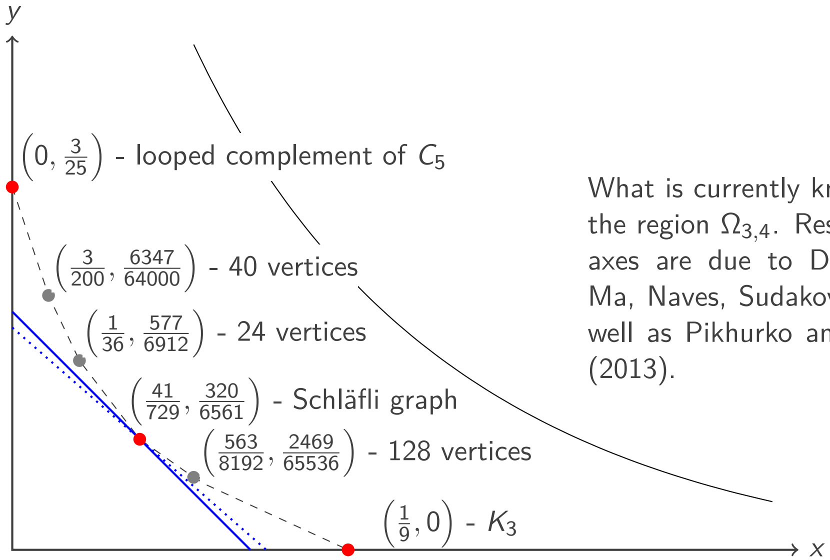

What is currently known about the region $\Omega_{3,4}$. Results on the axes are due to Das, Huang, Ma, Naves, Sudakov (2013) as well as Pikhurko and Vaughan (2013).

Thank you for your attention!