Proofs in Extremal Combinatorics through Optimization

6th RIKEN-IMI-ISM-NUS-ZIB-MODAL-NHR Workshop

Christoph Spiegel

17th of September 2022

Results are joint work with...

Olaf Parczyk
Freie Universität Berlin

Sebastian Pokutta
Zuse Institute Berlin

Tibor Szabó
Freie Universität Berlin

Research partially funded through Math+ project EF1-12

1. The Ramsey Multiplicity Problem
2. Search Heuristics for Upper Bounds
3. Flag Algebras for Lower Bounds
4. A Related Problem

The Ramsey Multiplicity of triangles

Theorem (Ramsey 1930)

For any $t \in \mathbb{N}$ there exists $R(t) \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R(t)$ contains a monochromatic clique of size t.

A well-known question: Can we determine $R(t)$?

A related question: How many cliques do we need to have? That means, letting $k_t(G)$ denote the fraction of all possible t-cliques in G, what is

$$c_t = \lim_{n\to\infty} \min\{k_t(\overline{G}) + k_t(G) : G \text{ graph of order } n\}?$$

Theorem (Goodman 1959)

$c_3 = 1/4.$

→ Same as Erdős-Rényi random graph! →

Conjecture (Erdős 1962)

$c_t = 2^{1-\binom{t}{2}}.$

The Ramsey Multiplicity of triangles

Theorem (Ramsey 1930)

For any $t \in \mathbb{N}$ there exists $R(t) \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R(t)$ contains a monochromatic clique of size t.

A well-known question: Can we determine $R(t)$?

A related question: How many cliques do we need to have? That means, letting $k_t(G)$ denote the fraction of all possible t-cliques in G, what is

$$c_t = \lim_{n\to\infty} \min\{k_t(\overline{G}) + k_t(G) : G \text{ graph of order } n\}?$$

Theorem (Goodman 1959)

$c_3 = 1/4.$

→ Same as Erdős-Rényi random graph! →

Conjecture (Erdős 1962)

$c_t = 2^{1-\binom{t}{2}}.$

The Ramsey Multiplicity of triangles

Theorem (Ramsey 1930)

For any $t \in \mathbb{N}$ there exists $R(t) \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R(t)$ contains a monochromatic clique of size t.

A well-known question: Can we determine $R(t)$?

A related question: How many cliques do we need to have? That means, letting $k_t(G)$ denote the fraction of all possible t-cliques in G, what is

$$c_t = \lim_{n \to \infty} \min\{k_t(\overline{G}) + k_t(G) : G \text{ graph of order } n\}?$$

Theorem (Goodman 1959)

$c_3 = 1/4$.

→ Same as Erdős-Rényi random graph! →

Conjecture (Erdős 1962)

$c_t = 2^{1-\binom{t}{2}}$.

The Ramsey Multiplicity of triangles

Theorem (Ramsey 1930)

For any $t \in \mathbb{N}$ there exists $R(t) \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R(t)$ contains a monochromatic clique of size t.

A well-known question: Can we determine $R(t)$?

A related question: How many cliques do we need to have? That means, letting $k_t(G)$ denote the fraction of all possible t-cliques in G, what is

$$c_t = \lim_{n\to\infty} \min\{k_t(\overline{G}) + k_t(G) : G \text{ graph of order } n\}?$$

Theorem (Goodman 1959)

$c_3 = 1/4$.

→ Same as Erdős-Rényi random graph! →

Conjecture (Erdős 1962)

$c_t = 2^{1-\binom{t}{2}}$.

The Ramsey Multiplicity of triangles

Theorem (Ramsey 1930)

For any $t \in \mathbb{N}$ there exists $R(t) \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R(t)$ contains a monochromatic clique of size t.

A well-known question: Can we determine $R(t)$?

A related question: How many cliques do we need to have? That means, letting $k_t(G)$ denote the fraction of all possible t-cliques in G, what is

$$c_t = \lim_{n \to \infty} \min\{k_t(\overline{G}) + k_t(G) : G \text{ graph of order } n\}?$$

Theorem (Goodman 1959)

$c_3 = 1/4$.

→ Same as Erdős-Rényi random graph! →

Conjecture (Erdős 1962)

$c_t = 2^{1-\binom{t}{2}}$.

The Ramsey Multiplicity of triangles

Theorem (Ramsey 1930)

For any $t \in \mathbb{N}$ there exists $R(t) \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R(t)$ contains a monochromatic clique of size t.

A well-known question: Can we determine $R(t)$?

A related question: How many cliques do we need to have? That means, letting $k_t(G)$ denote the fraction of all possible t-cliques in G, what is

$$c_t = \lim_{n\to\infty} \min\{k_t(\overline{G}) + k_t(G) : G \text{ graph of order } n\}?$$

Theorem (Goodman 1959)

$c_3 = 1/4$.

→ Same as Erdős-Rényi random graph! →

Conjecture (Erdős 1962)

$c_t = 2^{1-\binom{t}{2}}$.

The Ramsey Multiplicity of triangles

Theorem (Ramsey 1930)

For any $t \in \mathbb{N}$ there exists $R(t) \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R(t)$ contains a monochromatic clique of size t.

A well-known question: Can we determine $R(t)$?

A related question: How many cliques do we need to have? That means, letting $k_t(G)$ denote the fraction of all possible t-cliques in G, what is

$$c_t = \lim_{n\to\infty} \min\{k_t(\overline{G}) + k_t(G) : G \text{ graph of order } n\}?$$

Theorem (Goodman 1959)

$c_3 = 1/4$.

→ Same as Erdős-Rényi random graph! →

Conjecture (Erdős 1962)

$c_t = 2^{1-\binom{t}{2}}$.

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1989)

$c_4 \leq 0.976 \cdot 2^{-5}$ *and* $c_5 \leq 0.906 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1997)

$c_4 \leq 0.970 \cdot 2^{-5}$ *and* $c_5 \leq 0.881 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1997)

$c_4 \leq 0.970 \cdot 2^{-5}$ *and* $c_5 \leq 0.881 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1997)

$c_4 \leq 0.970 \cdot 2^{-5}$ *and* $c_5 \leq 0.881 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1997)

$c_4 \leq 0.970 \cdot 2^{-5}$ *and* $c_5 \leq 0.881 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1997)

$c_4 \leq 0.970 \cdot 2^{-5}$ *and* $c_5 \leq 0.881 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1997)

$c_4 \leq 0.970 \cdot 2^{-5}$ *and* $c_5 \leq 0.881 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1997)

$c_4 \leq 0.970 \cdot 2^{-5}$ *and* $c_5 \leq 0.881 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1997)

$c_4 \leq 0.970 \cdot 2^{-5}$ *and* $c_5 \leq 0.881 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Ramsey Multiplicity beyond triangles

Theorem (Thomason 1997)

$c_4 \leq 0.970 \cdot 2^{-5}$ *and* $c_5 \leq 0.881 \cdot 2^{-9}$.

Theorem (Even-Zohar and Linial '15)

$c_4 \leq 0.969 \cdot 2^{-5}$.

Erdős conjecture was false! But what about lower bounds?

Theorem (Giraud 1976)

$c_4 \geq 0.695 \cdot 2^{-5}$.

Theorem (Sperfeld / Nieß'11)

$c_4 \geq 0.914 \cdot 2^{-5}$.

Theorem (Grzesik et al. '20)

$c_4 \geq 0.947 \cdot 2^{-5}$.

Both the best upper and lower bounds heavily rely on computer-assistence!

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_4 \leq 0.964 \cdot 2^{-5}$ and $0.780 \cdot 2^{-9} \leq c_5 \leq 0.874 \cdot 2^{-9}$.

How can we use Optimization to formulate mathematical proofs?

Graph blow-ups

We want constructive bounds that are ‘finitely describable’. Random graphs are one source for such constructions. Another natural deterministic one are graph blow-ups.

Definition

The *m-fold blow-up* $C[m]$ of a graph C is given by replacing each vertex in C with an independent set of size m. Two vertices are adjacent if the originals were.

Using blow-ups, we can derive an upper bounds for c_t from **any** graph C through

$$c_t \leq \lim_{m\to\infty} k_t(\overline{C[m]}) + k_t(C[m]). \tag{1}$$

This is in fact efficiently computable since

$$\lim_{m\to\infty} k_t(C[m]) = n^{\underline{t}}\, k_t(C) \;/ n^t \quad \text{and} \quad \lim_{m\to\infty} k_t(\overline{C[m]}) = \sum_{j=1}^{t} S(t,j) n^{\underline{j}}\, k_j(\overline{C}) / n^t. \tag{2}$$

Graph blow-ups

We want constructive bounds that are ‘finitely describable’. Random graphs are one source for such constructions. Another natural deterministic one are graph blow-ups.

Definition

The *m-fold blow-up* $C[m]$ of a graph C is given by replacing each vertex in C with an independent set of size m. Two vertices are adjacent if the originals were.

Using blow-ups, we can derive an upper bounds for c_t from **any** graph C through

$$c_t \leq \lim_{m \to \infty} k_t(\overline{C[m]}) + k_t(C[m]). \tag{1}$$

This is in fact efficiently computable since

$$\lim_{m \to \infty} k_t(C[m]) = n^{\underline{t}}\, k_t(C) \,/ n^t \quad \text{and} \quad \lim_{m \to \infty} k_t(\overline{C[m]}) = \sum_{j=1}^{t} S(t,j) n^{\underline{j}}\, k_j(\overline{C}) / n^t. \tag{2}$$

Graph blow-ups

We want constructive bounds that are 'finitely describable'. Random graphs are one source for such constructions. Another natural deterministic one are graph blow-ups.

Definition

The *m-fold blow-up* $C[m]$ of a graph C is given by replacing each vertex in C with an independent set of size m. Two vertices are adjacent if the originals were.

Using blow-ups, we can derive an upper bounds for c_t from **any** graph C through

$$c_t \leq \lim_{m\to\infty} k_t(\overline{C[m]}) + k_t(C[m]). \tag{1}$$

This is in fact efficiently computable since

$$\lim_{m\to\infty} k_t(C[m]) = n^t\, k_t(C)\ /n^t \quad \text{and} \quad \lim_{m\to\infty} k_t(\overline{C[m]}) = \sum_{j=1}^{t} S(t,j) n^j\, k_j(\overline{C})/n^t. \tag{2}$$

Graph blow-ups

We want constructive bounds that are 'finitely describable'. Random graphs are one source for such constructions. Another natural deterministic one are graph blow-ups.

Definition

The *m-fold blow-up* $C[m]$ of a graph C is given by replacing each vertex in C with an independent set of size m. Two vertices are adjacent if the originals were.

Using blow-ups, we can derive an upper bounds for c_t from **any** graph C through

$$c_t \leq \lim_{m\to\infty} k_t(\overline{C[m]}) + k_t(C[m]). \tag{1}$$

This is in fact efficiently computable since

$$\lim_{m\to\infty} k_t(C[m]) = n^t\, k_t(C) \,/ n^t \quad \text{and} \quad \lim_{m\to\infty} k_t(\overline{C[m]}) = \sum_{j=1}^{t} S(t,j) n^j\, k_j(\overline{C})/n^t. \tag{2}$$

Graph blow-ups

We want constructive bounds that are ‘finitely describable’. Random graphs are one source for such constructions. Another natural deterministic one are graph blow-ups.

Definition

The *m-fold blow-up* $C[m]$ of a graph C is given by replacing each vertex in C with an independent set of size m. Two vertices are adjacent if the originals were.

Using blow-ups, we can derive an upper bounds for c_t from **any** graph C through

$$c_t \leq \lim_{m \to \infty} k_t(\overline{C[m]}) + k_t(C[m]). \tag{1}$$

This is in fact efficiently computable since

$$\lim_{m \to \infty} k_t(C[m]) = n^{\underline{t}}\, k_t(C) \;/n^t \quad \text{and} \quad \lim_{m \to \infty} k_t(\overline{C[m]}) = \sum_{j=1}^{t} S(t,j) n^{\underline{j}}\, k_j(\overline{C})/n^t. \tag{2}$$

Constructing graphs through search heuristics

For fixed n and $\mathbf{s} \in \{0,1\}^{\binom{n}{2}}$ let $C_{\mathbf{s}} = \left([n], \{ij : i < j, s_{\binom{j-1}{2}+i} = 1\}\right)$ and consider

$$\min_{\mathbf{s} \in \{0,1\}^{\binom{n}{2}}} \sum_{j=1}^{s} \frac{S(t,j) n^{\underline{j}} k_j(\overline{C_{\mathbf{s}}})}{n^t} + \frac{n^{\underline{t}} k_t(C_{\mathbf{s}})}{n^t}.$$

So we have found our optimization problem! How to solve it?

Approach 1. For $n \lessapprox 7$ we can check all states $\mathbf{s}$ exhaustively.

Unfortunately even $n = 40$ is much too small for c_4 and c_5, barely disproving Erdős' original conjecture. **Can we use combinatorial insights to bias the search space?**

Constructing graphs through search heuristics

For fixed n and $\mathbf{s} \in \{0,1\}^{\binom{n}{2}}$ let $C_\mathbf{s} = \left([n], \{ij : i < j, s_{\binom{j-1}{2}+i} = 1\}\right)$ and consider

$$\min_{\mathbf{s} \in \{0,1\}^{\binom{n}{2}}} \sum_{j=1}^{s} \frac{S(t,j) n^{\underline{j}} k_j(\overline{C_\mathbf{s}})}{n^t} + \frac{n^{\underline{t}} k_t(C_\mathbf{s})}{n^t}.$$

So we have found our optimization problem! How to solve it?

Approach 1. For $n \lessapprox 7$ we can check all states $\mathbf{s}$ exhaustively.

Unfortunately even $n = 40$ is much too small for c_4 and c_5, barely disproving Erdős' original conjecture. **Can we use combinatorial insights to bias the search space?**

Constructing graphs through search heuristics

For fixed n and $\mathbf{s} \in \{0,1\}^{\binom{n}{2}}$ let $C_{\mathbf{s}} = \left([n], \{ij : i < j, s_{\binom{j-1}{2}+i} = 1\}\right)$ and consider

$$\min_{\mathbf{s} \in \{0,1\}^{\binom{n}{2}}} \sum_{j=1}^{s} \frac{S(t,j) n^{\underline{j}} k_j(\overline{C_{\mathbf{s}}})}{n^t} + \frac{n^{\underline{t}} k_t(C_{\mathbf{s}})}{n^t}.$$

So we have found our optimization problem! How to solve it?

Approach 2. For $n \lessapprox 10$ we can generate all graphs up to isomorphism.

Unfortunately even $n = 40$ is much too small for c_4 and c_5, barely disproving Erdős' original conjecture. **Can we use combinatorial insights to bias the search space?**

Constructing graphs through search heuristics

For fixed n and $\mathbf{s} \in \{0,1\}^{\binom{n}{2}}$ let $C_{\mathbf{s}} = \left([n], \{ij : i < j, s_{\binom{j-1}{2}+i} = 1\}\right)$ and consider

$$\min_{\mathbf{s}\in\{0,1\}^{\binom{n}{2}}} \sum_{j=1}^{s} \frac{S(t,j)\, n^{\underline{j}}\, k_j(\overline{C_{\mathbf{s}}})}{n^t} + \frac{n^{\underline{t}}\, k_t(C_{\mathbf{s}})}{n^t}.$$

So we have found our optimization problem! How to solve it?

Approach 3. For $n \lessapprox 15$ we can use a Bounded Search Tree.

Unfortunately even $n = 40$ is much too small for c_4 and c_5, barely disproving Erdős' original conjecture. **Can we use combinatorial insights to bias the search space?**

Constructing graphs through search heuristics

For fixed n and $\mathbf{s} \in \{0,1\}^{\binom{n}{2}}$ let $C_{\mathbf{s}} = \left([n], \{ij : i < j, s_{\binom{j-1}{2}+i} = 1\}\right)$ and consider

$$\min_{\mathbf{s}\in\{0,1\}^{\binom{n}{2}}} \sum_{j=1}^{s} \frac{S(t,j) n^{\underline{j}} k_j(\overline{C_{\mathbf{s}}})}{n^t} + \frac{n^{\underline{t}} k_t(C_{\mathbf{s}})}{n^t}.$$

So we have found our optimization problem! How to solve it?

Approach 4. For $n \lessapprox 40$ we can use Search Heuristics.

Unfortunately even $n = 40$ is much too small for c_4 and c_5, barely disproving Erdős' original conjecture. **Can we use combinatorial insights to bias the search space?**

Constructing graphs through search heuristics

For fixed n and $\mathbf{s} \in \{0,1\}^{\binom{n}{2}}$ let $C_\mathbf{s} = \left([n], \{ij : i < j, s_{\binom{j-1}{2}+i} = 1\}\right)$ and consider

$$\min_{\mathbf{s}\in\{0,1\}^{\binom{n}{2}}} \sum_{j=1}^{s} \frac{S(t,j)n^{\underline{j}}\, k_j(\overline{C_\mathbf{s}})}{n^t} + \frac{n^{\underline{t}}\, k_t(C_\mathbf{s})}{n^t}.$$

So we have found our optimization problem! How to solve it?

Approach 5? Good source of benchmark problems...

Unfortunately even $n = 40$ is much too small for c_4 and c_5, barely disproving Erdős' original conjecture. **Can we use combinatorial insights to bias the search space?**

Constructing graphs through search heuristics

For fixed n and $\mathbf{s} \in \{0,1\}^{\binom{n}{2}}$ let $C_{\mathbf{s}} = \left([n], \{ij : i < j, s_{\binom{j-1}{2}+i} = 1\}\right)$ and consider

$$\min_{\mathbf{s} \in \{0,1\}^{\binom{n}{2}}} \sum_{j=1}^{s} \frac{S(t,j) n^{\underline{j}} k_j(\overline{C_{\mathbf{s}}})}{n^t} + \frac{n^{\underline{t}} k_t(C_{\mathbf{s}})}{n^t}.$$

So we have found our optimization problem! How to solve it?

Approach 5? Good source of benchmark problems...

Unfortunately even $n = 40$ is much too small for c_4 and c_5, barely disproving Erdős' original conjecture. **Can we use combinatorial insights to bias the search space?**

Constructing graphs through search heuristics

For fixed n and $\mathbf{s} \in \{0,1\}^{\binom{n}{2}}$ let $C_{\mathbf{s}} = \left([n], \{ij : i < j, s_{\binom{j-1}{2}+i} = 1\}\right)$ and consider

$$\min_{\mathbf{s} \in \{0,1\}^{\binom{n}{2}}} \sum_{j=1}^{s} \frac{S(t,j)\, n^{\underline{j}}\, k_j(\overline{C_{\mathbf{s}}})}{n^t} + \frac{n^{\underline{t}}\, k_t(C_{\mathbf{s}})}{n^t}.$$

So we have found our optimization problem! How to solve it?

Approach 5? Good source of benchmark problems...

Unfortunately even $n = 40$ is much too small for c_4 and c_5, barely disproving Erdős' original conjecture. **Can we use combinatorial insights to bias the search space?**

Constructing Cayley graphs through search heuristics

Thomason's constructions are based on computing the values of XOR-graph-products. The results are in fact Cayley graphs in $C_3^{\times 2} \times C_2^{\times 5}$ and $C_3 \times C_2^{\times 6}$.

Definition

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Idea. Why not directly search Cayley graph constructions?

The binary vector $\mathbf{s}$ now represents the generating set S. Since $|G|/2 < |S| < |G|$ the number of variables is therefore linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for c_4 and c_5.

Constructing Cayley graphs through search heuristics

Thomason's constructions are based on computing the values of XOR-graph-products. The results are in fact Cayley graphs in $C_3^{\times 2} \times C_2^{\times 5}$ and $C_3 \times C_2^{\times 6}$.

Definition

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Idea. Why not directly search Cayley graph constructions?

The binary vector $\mathbf{s}$ now represents the generating set S. Since $|G|/2 < |S| < |G|$ the number of variables is therefore linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for c_4 and c_5.

Constructing Cayley graphs through search heuristics

Thomason's constructions are based on computing the values of XOR-graph-products. The results are in fact Cayley graphs in $C_3^{\times 2} \times C_2^{\times 5}$ and $C_3 \times C_2^{\times 6}$.

Definition

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Idea. Why not directly search Cayley graph constructions?

The binary vector $\mathbf{s}$ now represents the generating set S. Since $|G|/2 < |S| < |G|$ the number of variables is therefore linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for c_4 and c_5.

Constructing Cayley graphs through search heuristics

Thomason's constructions are based on computing the values of XOR-graph-products. The results are in fact Cayley graphs in $C_3^{\times 2} \times C_2^{\times 5}$ and $C_3 \times C_2^{\times 6}$.

Definition

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Idea. Why not directly search Cayley graph constructions?

The binary vector $\mathbf{s}$ now represents the generating set S. Since $|G|/2 < |S| < |G|$ the number of variables is therefore linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for c_4 and c_5.

Constructing Cayley graphs through search heuristics

Thomason's constructions are based on computing the values of XOR-graph-products. The results are in fact Cayley graphs in $C_3^{\times 2} \times C_2^{\times 5}$ and $C_3 \times C_2^{\times 6}$.

Definition

Given an abelian group G and set $S \subseteq G^\star$ satisfying $S^{-1} = S$, the associated *Cayley graph* has vertex set G and $g_1, g_2 \in G$ are adjacent if and only if $g_1^{-1} g_2 \in S$.

Idea. Why not directly search Cayley graph constructions?

The binary vector $\mathbf{s}$ now represents the generating set S. Since $|G|/2 < |S| < |G|$ the number of variables is therefore linear (instead of quadratic) in the number of vertices!

The groups $C_3 \times C_2^{\times 8}$ and $C_3 \times C_2^{\times 6}$ give the improved upper bounds for c_4 and c_5.

Proofs in Combinatorics through Optimization

1. The Ramsey Multiplicity Problem
2. Search Heuristics for Upper Bounds
3. **Flag Algebras for Lower Bounds**
4. A Related Problem

A trivial computational lower bound

The Flag Algebra SDP approach can be seen as (i) a formalized Cauchy-Schwarz-type argument and (ii) an improvement over a trivial computational lower bound.

Let $d_H(G)$ denote the probability that $v(H)$ vertices chosen uniformly at random in G induce a copy of H. Writing $c_t(G) = k_t(G) + k_t(\overline{G})$, basic double counting gives us

$$c_t(G) = \sum_{\substack{H \text{ graph} \\ v(H)=N}} d_H(G)\, c_t(H) \tag{3}$$

for $t \leq N \leq v(G)$. For any $N \geq t$ this implies a trivial lower bound of

$$c_t \geq \min_{\substack{H \text{ graph} \\ v(H)=N}} c_t(H). \tag{4}$$

A trivial computational lower bound

The Flag Algebra SDP approach can be seen as (i) a formalized Cauchy-Schwarz-type argument and **(ii) an improvement over a trivial computational lower bound.**

Let $d_H(G)$ denote the probability that $v(H)$ vertices chosen uniformly at random in G induce a copy of H. Writing $c_t(G) = k_t(G) + k_t(\overline{G})$, basic double counting gives us

$$c_t(G) = \sum_{\substack{H \text{ graph} \\ v(H)=N}} d_H(G)\, c_t(H) \tag{3}$$

for $t \leq N \leq v(G)$. For any $N \geq t$ this implies a trivial lower bound of

$$c_t \geq \min_{\substack{H \text{ graph} \\ v(H)=N}} c_t(H). \tag{4}$$

A trivial computational lower bound

The Flag Algebra SDP approach can be seen as (i) a formalized Cauchy-Schwarz-type argument and **(ii) an improvement over a trivial computational lower bound.**

Let $d_H(G)$ denote the probability that $v(H)$ vertices chosen uniformly at random in G induce a copy of H. Writing $c_t(G) = k_t(G) + k_t(\overline{G})$, basic double counting gives us

$$c_t(G) = \sum_{\substack{H \text{ graph} \\ v(H)=N}} d_H(G)\, c_t(H) \tag{3}$$

for $t \leq N \leq v(G)$. For any $N \geq t$ this implies a trivial lower bound of

$$c_t \geq \min_{\substack{H \text{ graph} \\ v(H)=N}} c_t(H). \tag{4}$$

A trivial computational lower bound

The Flag Algebra SDP approach can be seen as (i) a formalized Cauchy-Schwarz-type argument and **(ii) an improvement over a trivial computational lower bound.**

Let $d_H(G)$ denote the probability that $v(H)$ vertices chosen uniformly at random in G induce a copy of H. Writing $c_t(G) = k_t(G) + k_t(\overline{G})$, basic double counting gives us

$$c_t(G) = \sum_{\substack{H \text{ graph} \\ v(H)=N}} d_H(G)\, c_t(H) \tag{3}$$

for $t \leq N \leq v(G)$. For any $N \geq t$ this implies a trivial lower bound of

$$c_t \geq \min_{\substack{H \text{ graph} \\ v(H)=N}} c_t(H). \tag{4}$$

The Flag Algebras SDP approach

Razborov (2007) introduced *Flag Algebras* in order to study this type of problem. One important observation is that for any $Q \succeq 0$ the coefficients $a_H = \langle Q, D_H \rangle$ satisfy

$$\sum_{\substack{H \text{ graph} \\ v(H)=N}} d_H(G)\, a_H \leq O(1/v(G)) \tag{5}$$

for any graph G. Through (3) this implies the (hopefully improved) bound

$$c_t \geq \min_{\substack{H \text{ graph} \\ v(H)=N}} c_t(H) - a_H. \tag{6}$$

This approach gives the best current lower bounds for c_4 and c_5. The biggest bottleneck for further improvements consists of finding Q for larger N.

The Flag Algebras SDP approach

Razborov (2007) introduced *Flag Algebras* in order to study this type of problem. One important observation is that for any $Q \succeq 0$ the coefficients $a_H = \langle Q, D_H \rangle$ satisfy

$$\sum_{\substack{H \text{ graph} \\ v(H)=N}} d_H(G)\, a_H \leq O(1/v(G)) \tag{5}$$

for any graph G. Through (3) this implies the (hopefully improved) bound

$$c_t \geq \min_{\substack{H \text{ graph} \\ v(H)=N}} c_t(H) - a_H. \tag{6}$$

This approach gives the best current lower bounds for c_4 and c_5. The biggest bottleneck for further improvements consists of finding Q for larger N.

The Flag Algebras SDP approach

Razborov (2007) introduced *Flag Algebras* in order to study this type of problem. One important observation is that for any $Q \succeq 0$ the coefficients $a_H = \langle Q, D_H \rangle$ satisfy

$$\sum_{\substack{H \text{ graph} \\ v(H)=N}} d_H(G)\, a_H \leq O(1/v(G)) \tag{5}$$

for any graph G. Through (3) this implies the (hopefully improved) bound

$$c_t \geq \min_{\substack{H \text{ graph} \\ v(H)=N}} c_t(H) - a_H. \tag{6}$$

This approach gives the best current lower bounds for c_4 and c_5. The biggest bottleneck for further improvements consists of finding Q for larger N.

The Flag Algebras SDP approach

Razborov (2007) introduced *Flag Algebras* in order to study this type of problem. One important observation is that for any $Q \succeq 0$ the coefficients $a_H = \langle Q, D_H \rangle$ satisfy

$$\sum_{\substack{H \text{ graph} \\ v(H)=N}} d_H(G)\, a_H \leq O(1/v(G)) \tag{5}$$

for any graph G. Through (3) this implies the (hopefully improved) bound

$$c_t \geq \min_{\substack{H \text{ graph} \\ v(H)=N}} c_t(H) - a_H. \tag{6}$$

This approach gives the best current lower bounds for c_4 and c_5. The biggest bottleneck for further improvements consists of finding Q for larger N.

The Flag Algebras SDP approach

Razborov (2007) introduced *Flag Algebras* in order to study this type of problem. One important observation is that for any $Q \succeq 0$ the coefficients $a_H = \langle Q, D_H \rangle$ satisfy

$$\sum_{\substack{H \text{ graph} \\ v(H)=N}} d_H(G)\, a_H \leq O(1/v(G)) \tag{5}$$

for any graph G. Through (3) this implies the (hopefully improved) bound

$$c_t \geq \min_{\substack{H \text{ graph} \\ v(H)=N}} c_t(H) - a_H. \tag{6}$$

This approach gives the best current lower bounds for c_4 and c_5. The biggest bottleneck for further improvements consists of finding Q for larger N.

Proofs in Combinatorics through Optimization

Off-diagonal Ramsey Multiplicity

Question. Determining c_3 is easy, but even c_4 has been unresolved for over 60 years, so can we say more when studying the off-diagonal variant

$$c_{s,t} = \lim_{n \to \infty} \min\{k_s(\overline{G}) + k_t(G) : |G| = n\}?$$

A famous result of Reiher from 2016 implies that $c_{2,t} = 1/(t-1)$.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_{3,4} = 689 \cdot 3^{-8}$ and any large enough graph G admits a strong homomorphism into the Schläfli graph after changing at most $O(k_3(\overline{G}) + k_4(G) - c_{3,4})\, v(G)^2$ edges.

The fact that we can show stability proves that the search heuristic found a unique global optimum over all graphs of order 27!

Off-diagonal Ramsey Multiplicity

Question. Determining c_3 is easy, but even c_4 has been unresolved for over 60 years, so can we say more when studying the off-diagonal variant

$$c_{s,t} = \lim_{n\to\infty} \min\{k_s(\overline{G}) + k_t(G) : |G| = n\}?$$

A famous result of Reiher from 2016 implies that $c_{2,t} = 1/(t-1)$.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_{3,4} = 689 \cdot 3^{-8}$ and any large enough graph G admits a strong homomorphism into the Schläfli graph after changing at most $O(k_3(\overline{G}) + k_4(G) - c_{3,4})\, v(G)^2$ edges.

The fact that we can show stability proves that the search heuristic found a unique global optimum over all graphs of order 27!

Off-diagonal Ramsey Multiplicity

Question. Determining c_3 is easy, but even c_4 has been unresolved for over 60 years, so can we say more when studying the off-diagonal variant

$$c_{s,t} = \lim_{n\to\infty} \min\{k_s(\overline{G}) + k_t(G) : |G| = n\}?$$

A famous result of Reiher from 2016 implies that $c_{2,t} = 1/(t-1)$.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_{3,4} = 689 \cdot 3^{-8}$ and any large enough graph G admits a strong homomorphism into the Schläfli graph after changing at most $O(k_3(\overline{G}) + k_4(G) - c_{3,4})\, v(G)^2$ edges.

The fact that we can show stability proves that the search heuristic found a unique global optimum over all graphs of order 27!

Off-diagonal Ramsey Multiplicity

Question. Determining c_3 is easy, but even c_4 has been unresolved for over 60 years, so can we say more when studying the off-diagonal variant

$$c_{s,t} = \lim_{n\to\infty} \min\{k_s(\overline{G}) + k_t(G) : |G| = n\}?$$

A famous result of Reiher from 2016 implies that $c_{2,t} = 1/(t-1)$.

Theorem (Parczyk, Pokutta, S., and Szabó 2022+)

$c_{3,4} = 689 \cdot 3^{-8}$ and any large enough graph G admits a strong homomorphism into the Schläfli graph after changing at most $O(k_3(\overline{G}) + k_4(G) - c_{3,4})\, v(G)^2$ edges.

The fact that we can show stability proves that the search heuristic found a unique global optimum over all graphs of order 27!

ZUSE INSTITUTE BERLIN

Thank you for your attention!