# Leveraging Combinatorial Symmetries in Flag Algebra-based SDP Formulations

## Workshop on Recent Advances in Optimization, Fields Institute, CA

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# Results are joint work with...

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FU Berlin

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# Combinatorial Symmetries in Flag Algebras

1. What are we interested in?
2. Where does Optimization come in?
3. Where can we go from here?

# The Ramsey Multiplicity Problem

**Theorem (Ramsey 1930)**

*For any $t \in \mathbb{N}$ there exists $R_{t,t} \in \mathbb{N}$ such that any 2-edge-coloring of the complete graph of order at least $R_{t,t}$ contains a monochromatic clique of size $t$.*

**A well-known question:**

Can we determine $R_{t_1,\dots,t_c}$?

**A related question:**

*How many* cliques do we need to have?

Letting $\chi : E(K_n) \to [c]$ denote a coloring of the complete graph, $\chi_i$ the graph given by color $i$ and $k_{t_i}(\chi_i)$ count all $t_i$-cliques in that color, we want to determine

$$m_{t_1,\dots,t_c}(n) = \min_{\chi} m_{t_1,\dots,t_c}(\chi) = \min_{\chi} \frac{k_{t_1}(\chi_1)}{\binom{n}{t_1}} + \dots \frac{k_{t_c}(\chi_c)}{\binom{n}{t_c}}.$$

# The Ramsey Multiplicity Problem

**Theorem (Ramsey 1930 – Multicolor Version)**

*For any $t_1, \ldots, t_c \in \mathbb{N}$ there exists $R_{t_1,\ldots,t_c} \in \mathbb{N}$ s.t. any $c$-edge-coloring of $K_n$ with $n \geq R_{t_1,\ldots,t_c} \in \mathbb{N}$ contains an clique of size $t_i$ with edges colored $i$ for some $1 \leq i \leq c$.*

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**Theorem (Ramsey 1930 – Multicolor Version)**

*For any $t_1, \dots, t_c \in \mathbb{N}$ there exists $R_{t_1,\dots,t_c} \in \mathbb{N}$ s.t. any c-edge-coloring of $K_n$ with $n \geq R_{t_1,\dots,t_c} \in \mathbb{N}$ contains an clique of size $t_i$ with edges colored $i$ for some $1 \leq i \leq c$.*

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# A connection to Ramsey Multiplicity Numbers

The two questions are related through the following inequality:

$$m_{t_1,\dots t_c} = \lim_{n\to\infty} m_{t_1,\dots t_c}(n) \leq (R_{t_1,\dots,t_{c-1}} - 1)^{1-t_c}. \tag{1}$$

**Proof** Write $R = R_{t_1,\dots,t_{c-1}}$ and let $\chi$ be an extremal $c-1$-coloring of $K_{R-1}$.

Consider the sequence of $c$-colorings $\chi_m$ of $K_{m(R-1)}$ obtained by replacing each $v \in [R-1]$ by $m$ copies $v_1, \dots, v_m$ and setting $\chi_m(v_i w_j) = \chi(vw)$. The edges $v_i v_j$ get colored with the additional color $c$.

The graphs $\chi_m^i$ avoid cliques of size $t_i$ for $1 \leq i \leq c-1$ and each of the $m$ large cliques in $\chi_m^c$ contains exactly an $(R-1)^{-t_c}$ proportion of all possible $t_c$-cliques. □

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**Proof Sketch** Blow up $\chi$ while using the additional color $c$ for the cliques. □

**Theorem (Goodman 1959)**

*Exact characterization of $m_{3,3}(n)$ for all $n \in \mathbb{N}$ implying $m_{3,3} = 1/4 = (R_3 - 1)^{1-3}$.*

**Proof Sketch** The upper bound follows from Equation (1). The matching lower bound follows through double counting argument and an application of Cauchy-Schwarz. □

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# Fixing two colors and growing the size of cliques

When $c = 2$ and $t_1 = t_2 = t$, we have $R_t = t$ and a uniform random coloring beats (1).

**Conjecture (Erdős 1962)**

$m_{t,t}(n) = 2^{1-\binom{t}{2}}$ *for any* $t \in \mathbb{N}$. ***False for*** $t \geq 4$ ***(Thomason 1989)***

**Theorem (Thomason 1989)**

$m_{4,4} \leq 0.03050$ *and* $m_{5,5} \leq 0.001770$.

**Theorem (Giraud 1976)**

$m_{4,4} \geq 0.02172$

**Theorem (Parczyk, Pokutta, S., Szabó 2022)**

$m_{3,4} = 689/3^8$ and the extremal constructions are based on the Schläfli graph.

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**Theorem (Even-Zohar, Linial 2016)**

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**Theorem (Parczyk, Pokutta, S., Szabó 2022)**

$m_{4,4} \leq 0.03013$ and $m_{5,5} \leq 0.001707$.

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**Theorem (Sperfeld / Niess 2011)**

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**Theorem (Kiem, Pokutta, S. 2022+)**

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# Fixing triangles and growing the number of colors

**Theorem (Goodman 1959)**

$m_{3,3} = 1/4 = 1/(R_3 - 1)^2$ *and there are many extremal constructions.*

**Theorem (Cummings et al. 2013)**

$m_{3,3,3} = 1/25 = 1/(R_{3,3} - 1)^2$ *and the only extremal constructions are based on* $R_{3,3}$.

**Theorem (Kiem, Pokutta, S. 2022+)**

$m_{3,3,3,3} = 1/256 = 1/(R_{3,3,3} - 1)^2$.

**Open Problem:** $m_{3,\ldots,3} = (R_{3,\ldots,3} - 1)^{-2}$ for all $c$ or $m_{3,\ldots,3} \cdot (R_{3,\ldots,3} - 1)^2 = o(1)$?

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# Combinatorial Symmetries in Flag Algebras

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2. Where does Optimization come in?
3. Where can we go from here?

# Upper bounds through Discrete Optimization

Using blow-ups, we can derive an upper bound for $m_{t_1,\ldots,t_c}$ from **any** coloring

$$\chi : E(K_n) \cup \{n\} \to [c]$$

of the looped complete graph through

$$m_{t_1,\ldots,t_c} \leq m'_{t_1,\ldots,t_c}(\chi) = \frac{k'_{t_1}(\chi_1)}{n^{t_1}} + \ldots + \frac{k'_{t_c}(\chi_c)}{n^{t_c}},$$

where $k'_{t_i}(\chi_i)$ counts *not necessarily injective* copies of $t_i$-cliques in $\chi_i$.

So we have found a discrete optimization problem! How to solve it?

**Approach 0.** Check some obvious existing constructions.

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where $k'_{t_i}(\chi_i)$ counts *not necessarily injective* copies of $t_i$-cliques in $\chi_i$.

So we have found a discrete optimization problem! How to solve it?

**Approach 1** Check colorings $\chi$ exhaustively.

# Upper bounds through Discrete Optimization

Using blow-ups, we can derive an upper bound for $m_{t_1,\ldots,t_c}$ from **any** coloring

$$\chi : E(K_n) \cup \{n\} \to [c]$$

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**Approach 1.5** Check colorings $\chi$ exhaustively *up to isomorphism*.

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**Approach 2** Bounded Tree search taking the parameter into account.

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**Approach 3** Search Heuristics $\rightarrow$ upper bounds for $m_{4,4}$, $m_{5,5}$ and $m_{3,4}$

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**Approach 4?** Good source of benchmark problems...

# Lower bounds through Semidefinite Programming

Razborov (2007) introduced *Flag Algebras* to study the limits of discrete objects. They allow one to apply a Cauchy-Schwarz-argument by solving a concrete SDP formulation.

Can be seen as an improvement over a trivial computational bound:

Letting $d_\varphi(\chi)$ denote the density of $\varphi$ in $\chi$, double counting gives us

$$m_{t_1,\dots,t_c}(\chi) = \sum_{\varphi \text{ of order } N} d_\varphi(\chi)\, m_{t_1,\dots,t_c}(\varphi)$$

for $t_1, \dots, t_c \leq N \leq v(\chi)$. This implies a trivial lower bound of

$$m_{t_1,\dots,t_c} \geq \min_{\varphi \text{ of order } N} m_{t_1,\dots,t_c}(\varphi).$$

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Can be seen as an improvement over a trivial computational bound:

Letting $D_\varphi$ denote a particular, computable matrix of density values, any $Q \succeq 0$ satisfies

$$\sum_{\varphi} d_\varphi(\chi) \langle Q, D_\varphi \rangle \leq O(1/v(\chi))$$

for any coloring $\chi$. This implies the following strengthening:

$$m_{t_1,\dots,t_c} \geq \max_{\boldsymbol{Q} \succeq \boldsymbol{0}} \min_{\varphi \text{ of order } N} m_{t_1,\dots,t_c}(\varphi) - \langle \boldsymbol{Q}, \boldsymbol{D}_\varphi \rangle \ . \tag{2}$$

# Leveraging Symmetries

Increasing $N$ both improves the bound and makes the SDP more difficult to solve:

| $N$ | *value* | *time* | *memory* |
|---|---|---|---|
| 6 | 0.02875 | 0.2s ±0.0 | 81.2MB ±24.7 |
| 7 | 0.02918 | 4.9s ±0.1 | 126.9MB ±26.3 |
| 8 | 0.02942 | 1.8h ±0.1 | 1.8GB ±0.0 |
| 9 | ??? | - | - |

Table: Complexity of SDP problem formulations for $m_{4,4}$ using CSDP

**How can we use combinatorial information to reduce these SDP formulations?**

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|---|---|---|---|---|---|
| $N$ | *value* | *time* | *memory* | *time* | *memory* |
| 6 | 0.02875 | 0.2s ±0.0 | 81.2MB ±24.7 | 0.2s ±0.0 | 69.8MB ±25.1 |
| 7 | 0.02918 | 4.9s ±0.1 | 126.9MB ±26.3 | 2.4s ±0.0 | 100.7MB ±28.0 |
| 8 | 0.02942 | 1.8h ±0.1 | 1.8GB ±0.0 | 0.3h ±0.0 | 642.3MB ±18.1 |
| 9 | **0.02961** | - | - | weeks | less than 1.5TB |

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Increasing $N$ both improves the bound and makes the SDP more difficult to solve:

| | | **baseline** | | **ours** | |
|---|---|---|---|---|---|
| $N$ | *value* | *time* | *memory* | *time* | *memory* |
| 4 | 0.0 | 0.06s ±0.0 | 79.39MB ±21.0 | 0.02s ±0.0 | 3.61 MB ±0.0 |
| 5 | 0.002938 | 1.5m ±0.0 | 1.1 GB ±0.0 | 1.1s ±0.0 | 86.9 MB ± 22.8 |
| 6 | **0.003906** | – | – | 1 day | less than 128GB |

Table: Complexity of SDP problem formulations for $m_{3,3,3,3}$ using CSDP

**How can we use combinatorial information to reduce these SDP formulations?**

# Leveraging Symmetries

How can we use combinatorial information to reduce these SDP formulations?

**Observation** The $D_\varphi$ have a *block-diagonal structure* with many *symmetries*.

$$\max_{Q \succeq 0} \min \left\{ 1 - \left\langle Q, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle, - \left\langle Q, \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix} \right\rangle, - \left\langle Q, \begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix} \right\rangle, 1 - \left\langle Q, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \right\},$$

Figure: The SDP formulation for $m_{3,3}$

**Method 1** Reduce the number of constraints and blocks by combining constraints.

**Method 2** Reduce the number of variables by block diagonalization.

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**Observation** The $D_\varphi$ have a *block-diagonal structure* with many *symmetries*.

$$\max_{Q \succeq 0} \min \left\{ 1 - \left\langle Q, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \right\rangle, -\left\langle Q, \begin{pmatrix} 1/6 & 1/3 \\ 1/3 & 1/6 \end{pmatrix} \right\rangle \right\},$$

Figure: The SDP formulation for $m_{3,3}$

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**Observation** The $D_\varphi$ have a *block-diagonal structure* with many *symmetries*.

$$\max_{x,y\geq 0} \min \left\{1 - \frac{x}{2} - \frac{y}{2}, -\frac{x}{2} + \frac{y}{6}\right\}.$$

Figure: The SDP formulation for $m_{3,3}$

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# Combinatorial Symmetries in Flag Algebras

1. What are we interested in?
2. Where does Optimization come in?
3. Where can we go from here?

# Outlook

We have...

**...optimized the computation and formulation of the SDP problems.**

Can we...

**...improve the solution process and extend the approach to new domains?**

Thank you for your attention!